Chapter 8

(AST301) Design and Analysis of Experiments II

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Lecture Outline

1. 8. Two-Level Fractional Factorial Designs
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   - 8.2 The one-half fraction of the 2^k design
   - 8.3 The one-quarter fraction of the 2^k design
   - 8.4 The general 2^{k-p} fractional factorial design

Section 1

8. Two-Level Fractional Factorial Designs

Subsection 1

8.1 Introduction

8.1 Introduction

- As the number of factors in a 2^k factorial design increases, the number of runs required for a complete replicate of the design rapidly outgrows the resources of most experimenters.
- For example, a complete replicate of the 2^6 design requires 64 runs. In this design only 6 of the 63 degrees of freedom correspond to main effects, and only 15 degrees of freedom correspond to two-factor interactions.
- There are only 21 degrees of freedom associated with effects that are likely to be of major interest. The remaining 42 degrees of freedom are associated with three-factor and higher interactions.

8.1 Introduction

- If the experimenter can reasonably assume that certain high-order interactions are negligible, information on the main effects and low-order interactions may be obtained by running only a fraction of the complete factorial experiment.
- A major use of fractional factorials is in **screening experiments** — experiments in which many factors are considered and the objective is to identify those factors (if any) that have large effects.
- The factors identified as important are then investigated more thoroughly in subsequent experiments.

8.1 Introduction

The successful use of fractional factorial designs is based on three key ideas:

1. **The sparsity of effects principle:** when there are several variables, the system will be driven primarily by some of the main effects and low order interactions
2. **The projection property:** Fractional factorial designs can be projected into larger designs in the subset of significant factors
3. **Sequential experimentation:** It is possible to combine the runs of two or more fractional factorials to assemble sequentially a larger design to estimate the factor effects and interactions of interest

Subsection 2

8.2 The one-half fraction of the 2^k design

8.2 The one-half fraction of the 2^k design

- Consider a situation in which three factors, each at two levels, are of interest, but the experimenters cannot afford to run all $2^3 = 8$ treatment combinations.
- They can, however, afford four runs.
- This suggests a **one-half fraction** of a 2^3 design.
- Because the design contains $2^{3-1} = 4$ treatment combinations, a one-half fraction of the 2^3 design is often called a 2^{3-1} **design**.

8.2 The one-half fraction of the 2^k design

- The four treatment combinations can be selected according to the plus sign of a factorial effect, which is known as a **generator** or a **word**.
- Thus, if ABC is the generator of the 2^{3-1} design, then we select the four treatment combinations $a, b, c,$ and abc as our one-half fraction.
- We call $I = ABC$ as the **defining relation** for the design.
- In general, defining relation will always be the set of all columns that are equal to the identity column I.

8.2 The one-half fraction of the 2^k design

■ **TABLE 8.1**

Plus and Minus Signs for the 2^3 Factorial Design

Treatment Combination	Factorial Effect							
	I	*A*	*B*	*C*	*AB*	*AC*	*BC*	*ABC*
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
c	+	−	−	+	+	−	−	+
abc	+	+	+	+	+	+	+	+
ab	+	+	+	−	+	−	−	−
ac	+	+	−	+	−	+	−	−
bc	+	−	+	+	−	−	+	−
(1)	+	−	−	−	+	+	+	−

8.2 The one-half fraction of the 2^k design

- The estimates of the main effects of A, B, and C are

$$[A] = \frac{1}{2}(a - b - c + abc)$$
$$[B] = \frac{1}{2}(-a + b - c + abc)$$
$$[C] = \frac{1}{2}(-a - b + c + abc)$$

- The estimates of the interactions AB, BC, and AC are

$$[BC] = \frac{1}{2}(a - b - c + abc)$$
$$[AC] = \frac{1}{2}(-a + b - c + abc)$$
$$[AB] = \frac{1}{2}(-a - b + c + abc)$$

8.2 The one-half fraction of the 2^k design

- Thus, $[A] = [BC], [B] = [AC]$, and $[C] = [AB]$ consequently, it is impossible to differentiate between A and BC, B and AC, and C and AB.
- In fact, when we estimate A, B, and C we are really estimating $A + BC, B + AC$, and $C + AB$. Two or more effects that have this property are called **aliases**.
- In our example, A and BC are aliases, B and AC are aliases, and C and AB are aliases.

8.2 The one-half fraction of the 2^k design

- We indicate this by the notation $[A] \to A + BC, [B] \to B + AC$, and $[C] \to C + AB$.
- The alias structure for this design may be easily determined by using the defining relation $I = ABC$.
- Multiplying any column (or effect) by the defining relation yields the aliases for that column (or effect).
- In our example, this yields that the alias of A is BC.

$$A.I = A.ABC = A^2BC = BC$$

- Similarly, we find that (B and AC) and (C and AB) are aliases

8.2 The one-half fraction of the 2^k design

- This one-half fraction, with $I = +ABC$, is usually called the **principal fraction**.
- Now suppose that we had chosen the other one-half fraction, that is, the treatment combinations associated with minus in the ABC column, known as the **complementary** or **alternate fraction**.
- The defining relation for this design is $I = -ABC$.
- The alternate fraction gives us

$$[A]' \to A - BC$$
$$[B]' \to B - AC$$
$$[C]' \to C - AB$$

8.2 The one-half fraction of the 2^k design

Thus, when we estimate A, B, and C with this particular fraction, we are really estimating $A - BC$, $B - AC$, and $C - AB$.

In practice, it does not matter which fraction is actually used. The two one-half fractions form a complete 2^3 design.

Suppose that after running one of the one-half fractions of the 2^3 design, the other fraction was also run. Thus, all eight runs associated with the full 2^3 are now available.

8.2 The one-half fraction of the 2^k design

- We may now obtain **de-aliased estimates** of all the effects by analyzing the eight runs as a full 2^3 design in two blocks of four runs each.
- This could also be done by adding and subtracting the linear combination of effects from the two individual fractions.

$$\frac{1}{2}([A] + [A]') = \frac{1}{2}(A + BC + A - BC) \to A$$
$$\frac{1}{2}([A] - [A]') = \frac{1}{2}(A + BC - A + BC) \to BC$$

8.2 The one-half fraction of the 2^k design

Thus, for all three pairs of linear combinations, we would obtain the following:

i	From $\frac{1}{2}([i] + [i]')$	From $\frac{1}{2}([i] - [i]')$
A	A	BC
B	B	AC
C	C	AB

Furthermore, by assembling the full 2^3 in this fashion with $I = +ABC$ in the first group of runs and $I = -ABC$ in the second, the 2^3 confounds ABC with blocks

Design resolution

- The preceding $2^3 - 1$ design is called a **resolution III design**
- In such a design, main effects are aliased with two-factor interactions.

  A design is of resolution R if no p-factor effect is aliased with another effect containing less than $R - p$ factors

- We usually employ a Roman numeral subscript to denote design resolution;
- Thus, the one-half fraction of the 2^3 design with the defining relation $I = ABC$ (or $I = -ABC$) is a 2^{3-1}_{III} design.

Design resolution

- Design resolutions describe how much the effects in a fractional factorial design are aliased with other effects.
- When we do a fractional factorial design, one or more of the effects are confounded, meaning they cannot be estimated separately from each other.
- Usually, we want to use a fractional factorial design with the highest possible resolution.

Design resolution

Resolution III designs.

- These are designs in which no main effects are aliased with any other main effect, but main effects are aliased with two-factor interactions and some two-factor interactions may be aliased with each other.
- The 2^{3-1} design in Table 8.1 is of resolution III (2^{3-1}_{III})

Design resolution

Resolution IV design.

- These are designs in which no main effect is aliased with any other main effect or with any two-factor interaction, but two-factor interactions are aliased with each other.
- A 2^{4-1} design with $I = ABCD$ is a resolution IV design (2^{4-1}_{IV})

Design resolution

Resolution V design.

- These are designs in which no main effect or two-factor interaction is aliased with any other main effect or two-factor interaction, but two-factor interactions are aliased with three-factor interactions.
- A 2^{5-1} design with $I = ABCDE$ is a resolution V design (2^{5-1}_{V})

Design resolution

- In general, the resolution of a two-level fractional factorial design is equal to the **shortest number of letters** in any word in the defining relation

Higher the resolution better the design (why?)

Design resolution

- We never want a resolution II design, because a design would alias two main effects. Thus minimum acceptable resolution in III.
- Resolution III designs have some main effects aliased to two-factor interactions. If we believe that only main effects are present and all interactions are negligible, then a resolution III design is sufficient for estimating main effects.
- Resolution III designs are called main effects design for this reason.
- If we believe that some two factor interactions may be non negligible but all three way and higher interactions are negligible then a resolution IV is sufficient for main effects.
- Low resolution fractional factorials are often used as screening designs, where we are trying to screen many factors to see if any of them has an effect. This is usually an early stage of investigation.

Subsection 3

Construction and Analysis of the One-Half Fraction

Construction and Analysis of the One-Half Fraction

A one-half fraction of the 2^k design of the highest resolution may be constructed by

- writing down a **basic design** consisting of the runs for a full 2^{k-1} factorial and
- then adding the the k^{th} factor by identifying its plus and minus levels with the plus and minus signs of the highest order interaction $ABC \cdots (K-1)$

Constructing one-half fractions

- E.g. the 2^{3-1}_{III} fractional factorial design of the highest resolution can be obtained by writing down the full 2^2 factorial as the basic design and then equating the factor C to AB

■ **TABLE 8.2**
The Two One-Half Fractions of the 2^3 Design

	Full 2^2 Factorial (Basic Design)		$2^{3-1}_{\text{III}}, I = ABC$			$2^{3-1}_{\text{III}}, I = -ABC$		
Run	A	B	A	B	$C = AB$	A	B	$C = -AB$
1	−	−	−	−	+	−	−	−
2	+	−	+	−	−	+	−	+
3	−	+	−	+	−	−	+	+
4	+	+	+	+	+	+	+	−

Constructing one-half fractions

- Note that any interaction effect could be used to generate the column for the kth factor.
- However, using any effect other than ABC....(K-1) will not produce a design of the highest possible resolution.
- Another way to view the construction of a one-half fraction is to partition the runs into two blocks with the highest order interaction $ABC \ldots K$ confounded. Each block is a 2^{k-1} fractional factorial design of the highest resolution.

Constructing one-half fractions

Sequences of fractional factorials

- Fractional factorial designs could be less expensive and efficient in experimentation, if the runs can be made sequentially
- If the interest is in investigating $k = 4$ factors (i.e. $2^4 = 16$ runs) then it is preferable to run a 2^{4-1}_{IV} fractional design, analyze the results, and then decide on the best set of runs to perform the next.
- If necessary we can always run the alternate fraction and get the complete 2^4 design, where both half-fractions are considered as blocks

Constructing one-half fractions

Projection of Fractions into Factorials

Any fractional factorial design of resolution R contains complete factorial designs (possibly replicated factorials) in any subset of $R-1$ factors.

- This is an important and useful concept
- Because the maximum possible resolution of a one-half fraction of the 2^k design is $R = k$, every 2^{k-1} design will project into a full factorial in any $(k-1)$ of the original k factors.
- Furthermore, a 2^{k-1} design may be projected into two replicates of a full factorial in any subset of $k-2$ factors, four replicates of a full factorial in any subset of $k-3$ factors, and so on.

EXAMPLE 8.1

- Consider the filtration rate experiment in Example 6.2. The original design is a single replicate of the 2^4 design.
- In that example, we found that the main effects A, C, and D and the interactions AC and AD were different from zero.
- We will now return to this experiment and simulate what would have happened if a half-fraction of the 2^4 design had been run instead of the full factorial.

EXAMPLE 8.1

■ **TABLE 6.10**
Pilot Plant Filtration Rate Experiment

Run Number	Factor *A*	*B*	*C*	*D*	Run Label	Filtration Rate (gal/h)
1	−	−	−	−	(1)	45
2	+	−	−	−	*a*	71
3	−	+	−	−	*b*	48
4	+	+	−	−	*ab*	65
5	−	−	+	−	*c*	68
6	+	−	+	−	*ac*	60
7	−	+	+	−	*bc*	80
8	+	+	+	−	*abc*	65
9	−	−	−	+	*d*	43
10	+	−	−	+	*ad*	100
11	−	+	−	+	*bd*	45
12	+	+	−	+	*abd*	104
13	−	−	+	+	*cd*	75
14	+	−	+	+	*acd*	86
15	−	+	+	+	*bcd*	70
16	+	+	+	+	*abcd*	96

EXAMPLE 8.1

■ **TABLE 6.12**

Factor Effect Estimates and Sums of Squares for the 2^4 Factorial in Example 6.2

Model Term	Effect Estimate	Sum of Squares	Percent Contribution
A	21.625	1870.56	32.6397
B	3.125	39.0625	0.681608
C	9.875	390.062	6.80626
D	14.625	855.563	14.9288
AB	0.125	0.0625	0.00109057
AC	−18.125	1314.06	22.9293
AD	16.625	1105.56	19.2911
BC	2.375	22.5625	0.393696
BD	−0.375	0.5625	0.00981515
CD	−1.125	5.0625	0.0883363
ABC	1.875	14.0625	0.245379
ABD	4.125	68.0625	1.18763
ACD	−1.625	10.5625	0.184307
BCD	−2.625	27.5625	0.480942
ABCD	1.375	7.5625	0.131959

EXAMPLE 8.1

We will use the 2^{4-1} design with $I = ABCD$.

■ **TABLE 8.3**
The 2_{IV}^{4-1} Design with the Defining Relation $I = ABCD$

	Basic Design					
Run	***A***	***B***	***C***	***D = ABC***	**Treatment Combination**	**Filtration Rate**
1	−	−	−	−	(1)	45
2	+	−	−	+	*ad*	100
3	−	+	−	+	*bd*	45
4	+	+	−	−	*ab*	65
5	−	−	+	+	*cd*	75
6	+	−	+	−	*ac*	60
7	−	+	+	−	*bc*	80
8	+	+	+	+	*abcd*	96

EXAMPLE 8.1

- Using the defining relation, we note that each main effect is aliased with a three-factor interaction; that is, $A = A^2BCD = BCD, B = AB^2CD = ACD, C = ABC^2D = ABD$, and $D = ABCD^2 = ABC$.
- Furthermore, every two-factor interaction is aliased with another two-factor interaction. These alias relationships are $AB = CD, AC = BD$, and $BC = AD$.
- The four main effects plus the three two-factor interaction alias pairs account for the seven degrees of freedom for the design.

EXAMPLE 8.1

$$
\begin{aligned}
[A] =& \frac{1}{4}(-45 + 100 - 45 + 65 - 75 \\
& + 60 - 80 + 96) = 19.00 \rightarrow A + BCD \\
[AB] =& \frac{1}{4}(45 - 100 - 45 + 65 + 75 - 60 - 80 + 96) \\
=& -1.00 \rightarrow AB + CD
\end{aligned}
$$

EXAMPLE 8.1

TABLE 8.4
Estimates of Effects and Aliases from Example 8.1[a]

Estimate	Alias Structure
$[A] = 19.00$	$[A] \rightarrow A + BCD$
$[B] = 1.50$	$[B] \rightarrow B + ACD$
$[C] = 14.00$	$[C] \rightarrow C + ABD$
$[D] = 16.50$	$[D] \rightarrow D + ABC$
$[AB] = -1.00$	$[AB] \rightarrow AB + CD$
$[AC] = -18.50$	$[AC] \rightarrow AC + BD$
$[AD] = 19.00$	$[AD] \rightarrow AD + BC$

[a]Significant effects are shown in boldface type.

EXAMPLE 8.1

- Because factor B is not significant, we may drop it from consideration. Consequently, we may project this design into a single replicate of the 2^3 design in factors A, C, and D.
- Based on the above analysis, we can now obtain a model to predict filtration rate over the experimental region. This model is

$$
\begin{aligned}
\hat{y} =& \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_3 x_3 + \hat{\beta}_4 x_4 + \hat{\beta}_{13} x_1 x_3 + \hat{\beta}_{14} x_1 x_4 \\
\hat{y} =& 70.75 + \left(\frac{19.00}{2}\right) x_1 + \left(\frac{14.00}{2}\right) x_3 + \left(\frac{16.50}{2}\right) x_4 \\
& + \left(\frac{-18.50}{2}\right) x_1 x_3 + \left(\frac{19.00}{2}\right) x_1 x_4
\end{aligned}
$$

EXAMPLE 8.1

- There are two large effects associated with two-factor interactions, $AC + BD$ and $AD + BC$.
- In Example 8.2, we used the fact that the main effect of B was negligible to tentatively conclude that the important interactions were AC and AD.
- However, we can always isolate the significant interaction by running the alternate fraction, given by $I = -ABCD$.

EXAMPLE 8.1

Run	Basic Design			$D = -ABC$	Treatment	Filtration Rate
	A	B	C			
1	$-$	$-$	$-$	$+$	d	43
2	$+$	$-$	$-$	$-$	a	71
3	$-$	$+$	$-$	$-$	b	48
4	$+$	$+$	$-$	$+$	abd	104
5	$-$	$-$	$+$	$-$	c	68
6	$+$	$-$	$+$	$+$	acd	86
7	$-$	$+$	$+$	$+$	bcd	70
8	$+$	$+$	$+$	$-$	abc	65

EXAMPLE 8.1

$$
\begin{aligned}
[A]' &= 24.25 \rightarrow A - BCD \\
[B]' &= 4.75 \rightarrow B - ACD \\
[C]' &= 5.75 \rightarrow C - ABD \\
[D]' &= 12.75 \rightarrow D - ABC \\
[AB]' &= 1.25 \rightarrow AB - CD \\
[AC]' &= -17.75 \rightarrow AC - BD \\
[AD]' &= 14.25 \rightarrow AD - BC
\end{aligned}
$$

EXAMPLE 8.1

i	From $\frac{1}{2}([i] + [i]')$	From $\frac{1}{2}([i] - [i]')$
A	$21.63 \rightarrow A$	$-2.63 \rightarrow BCD$
B	$3.13 \rightarrow B$	$-1.63 \rightarrow ACD$
C	$9.88 \rightarrow C$	$4.13 \rightarrow ABD$
D	$14.63 \rightarrow D$	$1.88 \rightarrow ABC$
AB	$0.13 \rightarrow AB$	$-1.13 \rightarrow CD$
AC	$-18.13 \rightarrow AC$	$-0.38 \rightarrow BD$
AD	$16.63 \rightarrow AD$	$2.38 \rightarrow BC$

Subsection 4

8.3 The one-quarter fraction of the 2^k design

8.3 The one-quarter fraction of the 2^k design

- One-quarter fraction of the 2^k design could be useful when the number of factors is large.
- This design contains 2^{k-2} runs and is usually called a 2^{k-2} **fractional factorial**.
- The construction of the 2^{k-2} fractional factorial design requires to write down the **basic design** with $k-2$ factors first and then two additional columns are constructed from appropriately chosen interactions involving the first $k-2$ factors.

8.3 The one-quarter fraction of the 2^k design

- The one-quarter fraction of the 2^k design has two generators, if P and Q are the two generators then their generalized interaction PQ also acts as a generator
- **Complete defining relation** is $I = P = Q = PQ$
- As an example, consider 2^{6-2} design with $I = ABCE$ and $I = BCDF$ as the design generators
  - The complete defining relation for this design is $I = ABCE = BCDF = ADEF(ABCE \times BCDF)$
  - This 2^{6-2} design is a resolution IV design, **why?**

8.3 The one-quarter fraction of the 2^k design

- For the 2^{6-2} design with $I = ABCE = BCDF = ADEF$, the main effects are aliased with two- and five-factor interactions, e.g.

$$A(I) = A(ABCE) = A(BCDF) = A(ADEF)$$
$$A = BCE = ABCDF = DEF$$

- When we estimate the main effect of A, we actually estimate $A + BCE + DEF + ABCDF$
- Similarly, when we estimate 2-factor interaction AB, we actually estimate $AB + CE + ACDF + BDEF$

8.3 The one-quarter fraction of the 2^k design

The complete alias structure of this design is

TABLE 8.8

Alias Structure for the 2^{6-2}_{IV} Design with $I = ABCE = BCDF = ADEF$

$A = BCE = DEF = ABCDF$	$AB = CE = ACDF = BDEF$
$B = ACE = CDF = ABDEF$	$AC = BE = ABDF = CDEF$
$C = ABE = BDF = ACDEF$	$AD = EF = BCDE = ABCF$
$D = BCF = AEF = ABCDE$	$AE = BC = DF = ABCDEF$
$E = ABC = ADF = BCDEF$	$AF = DE = BCEF = ABCD$
$F = BCD = ADE = ABCEF$	$BD = CF = ACDE = ABEF$
	$BF = CD = ACEF = ABDE$
$ABD = CDE = ACF = BEF$	
$ACD = BDE = ABF = CEF$	

8.3 The one-quarter fraction of the 2^k design

- To construct the design, first write down the **basic design**, which consists of the 16 runs for a full $2^{6-2} = 2^4$ design in A, B, C, and D.
- Then the two factors E and F are added by associating their plus and minus levels with the plus and minus signs of the interactions ABC and BCD, respectively. This procedure is shown below

8.3 The one-quarter fraction of the 2^k design

■ **TABLE 8.9**

Construction of the 2^{6-2}_{IV} Design with the Generators $I = ABCE$ and $I = BCDF$

	Basic Design					
Run	A	B	C	D	$E = ABC$	$F = BCD$
1	−	−	−	−	−	−
2	+	−	−	−	+	−
3	−	+	−	−	+	+
4	+	+	−	−	−	+
5	−	−	+	−	+	+
6	+	−	+	−	−	+
7	−	+	+	−	−	−
8	+	+	+	−	+	−
9	−	−	−	+	−	+
10	+	−	−	+	+	+
11	−	+	−	+	+	−
12	+	+	−	+	−	−
13	−	−	+	+	+	−
14	+	−	+	+	−	−
15	−	+	+	+	−	+
16	+	+	+	+	+	+

8.3 The one-quarter fraction of the 2^k design

- Another way to construct this design is to derive the four blocks of the 2^6 design with $ABCE$ and $BCDF$ confounded and then choose the block with treatment combinations that are positive on $ABCE$ and $BCDF$.
- This would be a 2^{6-2} fractional factorial with generating relations $I = ABCE$ and $I = BCDF$, and because both generators $ABCE$ and $BCDF$ are positive, this is the principal fraction.

8.3 The one-quarter fraction of the 2^k design

- For this 2^{6-2} design, there are three alternate fractions, which are

$$
\begin{aligned}
&I = ABCE, \;\; I = -BCDF; \\
&I = -ABCE, \;\; I = BCDF; \quad \text{and} \\
&I = -ABCE, \;\; I = -BCDF
\end{aligned}
$$

EXAMPLE 8.4

A quality improvement team has decided to use a designed experiment to study the injection molding process so that shrinkage can be reduced. The team decides to investigate six factors Mold temperature (A), Screw speed (B), Holding time (C), Cycle time (D), Gate size (E), and holding pressure (F) - each at two levels.

The objective of this experiment is of learning how each factor affects shrinkage and also, something about how the factors interact. The team decides to use the 16-run two-level fractional factorial design.

EXAMPLE 8.4

■ **TABLE 8.10**

A 2_{IV}^{6-2} Design for the Injection Molding Experiment in Example 8.4

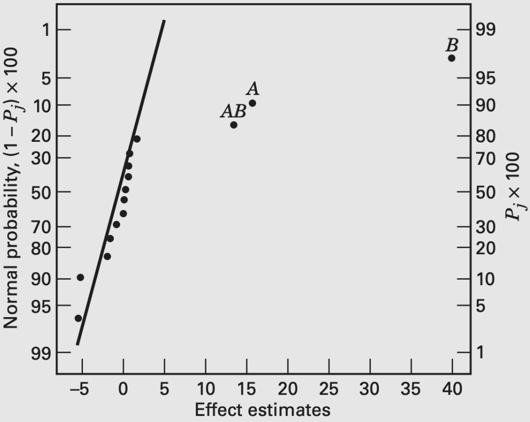
Run	Basic Design *A*	*B*	*C*	*D*	*E = ABC*	*F = BCD*	Observed Shrinkage (× 10)	
1	−	−	−	−	−	−	6	(1)
2	+	−	−	−	+	−	10	*ae*
3	−	+	−	−	+	+	32	*bef*
4	+	+	−	−	−	+	60	*abf*
5	−	−	+	−	+	+	4	*cef*
6	+	−	+	−	−	+	15	*acf*
7	−	+	+	−	−	−	26	*bc*
8	+	+	+	−	+	−	60	*abce*
9	−	−	−	+	−	+	8	*df*
10	+	−	−	+	+	+	12	*adef*
11	−	+	−	+	+	−	34	*bde*
12	+	+	−	+	−	−	60	*abd*
13	−	−	+	+	+	−	16	*cde*
14	+	−	+	+	−	−	5	*acd*
15	−	+	+	+	−	+	37	*bcdf*
16	+	+	+	+	+	+	52	*abcdef*

EXAMPLE 8.4

Variable[a]	Regression Coefficient	Estimated Effect	Sum of Squares
Overall Average	27.3125		
A	6.9375	13.8750	770.062
B	17.8125	35.6250	5076.562
C	−0.4375	−0.8750	3.063
D	0.6875	1.3750	7.563
E	0.1875	0.3750	0.563
F	0.1875	0.3750	0.563
$AB + CE$	5.9375	11.8750	564.063
$AC + BE$	−0.8125	−1.6250	10.562
$AD + EF$	−2.6875	−5.3750	115.562
$AE + BC + DF$	−0.9375	−1.8750	14.063
$AF + DE$	0.3125	0.6250	1.563
$BD + CF$	−0.0625	−0.1250	0.063
$BF + CD$	−0.0625	−0.1250	0.063
ABD	0.0625	0.1250	0.063
ABF	−2.4375	−4.8750	95.063

[a]Only main effects and two-factor interactions.

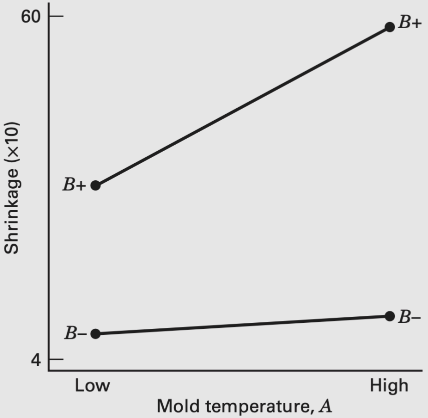
EXAMPLE 8.4



■ **FIGURE 8.12 Normal probability plot of effects for Example 8.4**

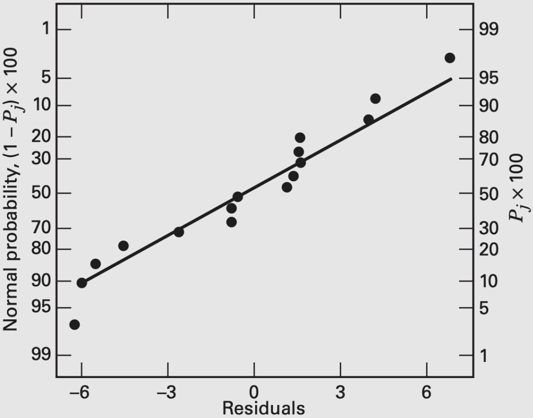
The only large effects are A (mold temperature), B (screw speed), and the AB interaction

EXAMPLE 8.4



■ **FIGURE 8.13** **Plot of *AB* (mold temperature-screw speed) interaction for Example 8.4**

EXAMPLE 8.4



■ FIGURE 8.14 Normal probability plot of residuals for Example 8.4

EXAMPLE 8.4

- The plot of the AB interaction shows that the process is very insensitive to temperature if the screw speed is at the low level but very sensitive to temperature if the screw speed is at the high level.
- Based on this initial analysis, the team decides to set both the mold temperature and the screw speed at the low level

Subsection 5

8.4 The general 2^{k-p} fractional factorial design

8.4 The general 2^{k-p} fractional factorial design

- A 2^k fractional design containing 2^{k-p} runs is called $1/2^p$ fraction of the 2^k design or 2^{k-p} **fractional factorial design**
- These designs require the selection of p independent generators and the corresponding defining relation consists of the initially chosen p generators and their $2^p - p - 1$ generalized interactions
- The alias structure can be found by multiplying each effect column by the defining relation and generators should be chosen carefully so that effects of potential interest are not aliased with each other, and each effect will have $2^p - 1$ aliases
- For moderately large values of k, we usually assume higher order interactions (say, third- or fourth-order and higher) to be negligible, and this greatly simplifies the alias structure.

8.4 The general 2^{k-p} fractional factorial design

- It is important to select the p generators for a 2^{k-p} fractional factorial design in a way that we obtain the best possible alias relationships
- A reasonable criterion is to select the generators such that the resulting 2^{k-p} design has the highest possible resolution, e.g. consider a 2^{6-2} design

Defining relation $I = ABCE = ABCDF \rightarrow$ Resolution III

Defining relation $I = ABCE = BCDF \rightarrow$ Resolution IV

8.4 The general 2^{k-p} fractional factorial design

- Sometimes resolution alone is insufficient to distinguish between designs. For example, consider the three ${2_{IV}}^{7-2}$ designs, which are based on different defining relations, but have the equal resolution IV.
- Sometimes resolution alone is insufficient to distinguish between designs. consider the three ${2_{IV}}^{7-2}$ designs given below

8.4 The general 2^{k-p} fractional factorial design

Design A Generators: $F = ABC, G = BCD$ $I = ABCF = BCDG = ADFG$	**Design B Generators:** $F = ABC, G = ADE$ $I = ABCF = ADEG = BCDEFG$	**Design C Generators:** $F = ABCD, G = ABDE$ $I = ABCDF = ABDEG = CEFG$
Aliases (two-factor interactions)	Aliases (two-factor interactions)	Aliases (two-factor interactions)
$AB = CF$	$AB = CF$	$CE = FG$
$AC = BF$	$AC = BF$	$CF = EG$
$AD = FG$	$AD = EG$	$CG = EF$
$AG = DF$	$AE = DG$	
$BD = CG$	$AF = BC$	
$BG = CD$	$AG = DE$	
$AF = BC = DG$		

8.4 The general 2^{k-p} fractional factorial design

- All of these designs are of resolution IV, but they have rather different alias structures (we have assumed that three-factor and higher interactions are negligible) with respect to the two-factor interactions.
- Clearly, design A has more extensive aliasing and design C the least, so design C would be the best choice for a ${2_{IV}}^{7-2}$.
- The three word lengths in design A are all 4; that is, the word length **pattern** is $\{4, 4, 4\}$.

8.4 The general 2^{k-p} fractional factorial design

- For design B it is $\{4, 4, 6\}$, and for design C it is $\{4, 5, 5\}$
- Notice that the defining relation for design C has only one four-letter word, whereas the other designs have two or three.
- Thus, design C minimizes the number of words in the defining relation that are of minimum length. We call such a design a **minimum aberration design**.

8.4 The general 2^{k-p} fractional factorial design

Minimizing aberration in a design of resolution R ensures that the design has the minimum number of main effects aliased with interactions of order $R-1$, the minimum number of two-factor interactions aliased with interactions of order $R-2$, and so forth.

Table 8.14 presents a selection of 2^{k-p} fractional factorial designs for $k \leq 15$ factors and up to $n \leq 128$ runs.

The suggested generators in this table will result in a design of the highest possible resolution. These are also the minimum aberration designs.

8.4 The general 2^{k-p} fractional factorial design

Number of Factors, k	Fraction	Number of Runs	Design Generators
3	2_{III}^{3-1}	4	$C = \pm AB$
4	2_{IV}^{4-1}	8	$D = \pm ABC$
5	2_{V}^{5-1}	16	$E = \pm ABCD$
	2_{III}^{5-2}	8	$D = \pm AB$
			$E = \pm AC$
6	2_{VI}^{6-1}	32	$F = \pm ABCDE$
	2_{IV}^{6-2}	16	$E = \pm ABC$
			$F = \pm BCD$
	2_{III}^{6-3}	8	$D = \pm AB$
			$E = \pm AC$
			$F = \pm BC$
7	2_{VII}^{7-1}	64	$G = \pm ABCDEF$
	2_{IV}^{7-2}	32	$F = \pm ABCD$
			$G = \pm ABDE$
	2_{IV}^{7-3}	16	$E = \pm ABC$
			$F = \pm BCD$
			$G = \pm ACD$

8.4 The general 2^{k-p} fractional factorial design

The alias relationships for all of the designs in Table 8.14 for which $n \leq 64$ are given in Appendix Table X(a-w).

The alias relationships presented in this table focus on main effects and two- and three-factor interactions. The complete defining relation is given for each design.

This appendix table makes it very easy to select a design of sufficient resolution to ensure that any interactions of potential interest can be estimated.

Analysis of 2^{k-p} fractional factorials

- The i^{th} effect is estimated by

$$l_i = \frac{\text{contrast}_i}{(N/2)},$$

where contrast_i can be found from the plus and minus sign of the column i and $N = 2^{k-p}$ is the total number of observations

- The 2^{k-p} design allows only $2^{k-p} - 1$ effects (and their aliases) to be estimated

Exercise 8.12.

An article in Industrial and Engineering Chemistry ("More on Planning Experiments to Increase Research Efficiency," 1970, pp. 60-65) uses a 2^{5-2} design to investigate the effect of $A =$ condensation temperature, $B =$ amount of material $1, C =$ solvent volume, $D =$ condensation time, and $E =$ amount of material 2 on yield. The results obtained are as Follows:

$$e = 23.2 \quad ad = 16.9 \quad cd = 23.8 \quad bde = 16.8$$
$$ab = 15.5 \quad bc = 16.2 \quad ace = 23.4 \quad \text{abcde} = 18.1$$

(a) Verify that the design generators used were $\mathrm{I} = ACE$ and $I = BDE$.

(b) Write down the complete defining relation and the aliases for this design.

(c) Estimate the main effects.

Exercise 8.12.

(d) Prepare an analysis of variance table. Verify that the AB and AD interactions are available to use as error.

(e) Plot the residuals versus the fitted values. Comment on the results.

Subsection 6

Blocking Fractional Factorials

Blocking Fractional Factorials

- Occasionally, a fractional factorial design requires so many runs that all of them cannot be made under homogeneous conditions.
- In these situations, fractional factorials may be confounded in blocks.
- Appendix Table X contains recommended blocking arrangements for many of the fractional factorial designs in Table 8.14. The minimum block size for these designs is eight runs

Blocking Fractional Factorials

- In assigning fractional factorials into blocks, we need to be careful about the effects to be confounded with blocks
- The 2^{6-2}_{IV} design with $I = ABCE = BCDF = ADEF$ is

$$(1), df, cef, bef, a, abef, acf, bc,$$
$$adef, bde, cde, abce, abd, acd, bcdf, abcdef,$$

- This fractional design contains 16 treatment combinations.
- We want to conduct the experiment into two blocks. Which effect should we select to be confounded with blocks?

Blocking Fractional Factorials

- In selecting an interaction to confound with blocks, we note from examining the alias structure in Appendix Table IX(f) that there are two alias sets involving only three-factor interactions.

(f) 2^{6-2}; 1/4 fraction of 6 factors in 16 runs **Resolution IV**

Design Generators

$E = ABC \quad F = BCD$

Defining relation: $I = ABCE = BCDF = ADEF$

Aliases

$A = BCE = DEF$	$AB = CE$
$B = ACE = CDF$	$AC = BE$
$C = ABE = BDF$	$AD = EF$
$D = BCF = AEF$	$AE = BC = DF$
$E = ABC = ADF$	$AF = DE$
$F = BCD = ADE$	$BD = CF$
$ABD = CDE = ACF = BEF$	$BF = CD$
$ACD = BDE = ABF = CEF$	

2 blocks of 8: $ABD = CDE = ACF = BEF$

- The table suggests selecting ABD (and its aliases) to be confounded with blocks.

Blocking Fractional Factorials

- Assume the effect ABD confounded with blocks, so the defining contrast $L = x_1 + x_2 + x_4 = 0 \pmod{2}$

$$(1), df, cef, bef, a, abef, acf, bc,$$
$$adef, bde, cde, abce, abd, acd, bcdf, abcdef,$$

Block-1 $(L = 0)$

$$(1), abef, adef, bde$$
$$abce, acd, bcdf, abcdef$$

Block-2 $(L = 1)$

$$df, cef, bef, a, acf,$$
$$bc, cde, abd$$

- Notice that the principal block contains those treatment combinations that have an even number of letters in common with ABD.

Example 8.6

A five-axis CNC machine is used to produce an impeller for a jet turbine engine. The blade profiles are an important quality characteristic.

Specifically, the deviation of the blade profile from the profile specified on the engineering drawing is of interest.

An experiment is run to determine which machine parameters affect profile deviation.

Example 8.6

The eight factors selected for the design are as follows:

$$
\begin{aligned}
A &= \text{ x-Axis shift} \\
B &= \text{ y-Axis shift} \\
C &= \text{ z-Axis shift} \\
D &= \text{ Tool supplier} \\
E &= \text{ a-Axis shift} \\
F &= \text{ Spindle speed } (\%) \\
G &= \text{ Fixture height} \\
H &= \text{ Feed rate } (\%)
\end{aligned}
$$

Example 8.6

The profile deviation is measured using a coordinate measuring machine, and the standard deviation of the difference between the actual profile and the specified profile is used as the response variable.

The machine has four spindles. Because there may be differences in the spindles, the process engineers feel that the spindles should be treated as blocks.

Example 8.6

The engineers feel confident that three-factor and higher interactions are not too important, but they are reluctant to ignore the two-factor interactions.

From Table 8.14, two designs initially appear appropriate: the 2_{IV}^{8-4} design with 16 runs and the 2_{IV}^{8-3} design with 32 runs. The experimenters decide to use the 2_{IV}^{8-3} design in four blocks.

8	2_{V}^{8-2}	64	$G = \pm ABCD$
			$H = \pm ABEF$
	2_{IV}^{8-3}	32	$F = \pm ABC$
			$G = \pm ABD$
			$H = \pm BCDE$
	2_{IV}^{8-4}	16	$E = \pm BCD$
			$F = \pm ACD$
			$G = \pm ABC$
			$H = \pm ABD$

Example 8.6

(l) 2^{8-4}; 1/16 fraction of 8 factors in 16 runs **Resolution IV**

Design Generators

$$E = BCD \quad F = ACD \quad G = ABC \quad H = ABD$$

Defining relation: $I = BCDE = ACDF = ABEF = ABCG = ADEG = BDFG = CEFG = ABDH$
$= ACEH = BCFH = DEFH = CDGH = BEGH = AFGH = ABCDEFGH$

Aliases

$A = CDF = BEF = BCG = DEG = BDH = CEH = FGH$	$AB = EF = CG = DH$
$B = CDE = AEF = ACG = DFG = ADH = CFH = EGH$	$AC = DF = BG = EH$
$C = BDE = ADF = ABG = EFG = AEH = BFH = DGH$	$AD = CF = EG = BH$
$D = BCE = ACF = AEG = BFG = ABH = EFH = CGH$	$AE = BF = DG = CH$
$E = BCD = ABF = ADG = CFG = ACH = DFH = BGH$	$AF = CD = BE = GH$
$F = ACD = ABE = BDG = CEG = BCH = DEH = AGH$	$AG = BC = DE = FH$
$G = ABC = ADE = BDF = CEF = CDH = BEH = AFH$	$AH = BD = CE = FG$
$H = ABD = ACE = BCF = DEF = CDG = BEG = AFG$	

2 blocks of 8: $AB = EF = CG = DH$

(m) 2^{8-3}; 1/8 fraction of 8 factors in 32 runs **Resolution IV**

Design Generators

$$F = ABC \quad G = ABD \quad H = BCDE$$

Defining relation: $I = ABCF = ABDG = CDFG = BCDEH = ADEFH = ACEGH = BEFGH$

Aliases

$A = BCF = BDG$	$AE = DFH = CGH$	$DE = BCH = AFH$
$B = ACF = ADG$	$AF = BC = DEH$	$DH = BCE = AEF$
$C = ABF = DFG$	$AG = BD = CEH$	$EF = ADH = BGH$
$D = ABG = CFG$	$AH = DEF = CEG$	$EG = ACH = BFH$
$E =$	$BE = CDH = FGH$	$EH = BCD = ADF = ACG = BFG$
$F = ABC = CDG$	$BH = CDE = EFG$	$FH = ADE = BEG$
$G = ABD = CDF$	$CD = FG = BEH$	$GH = ACE = BEF$
$H =$	$CE = BDH = AGH$	$ABE = CEF = DEG$
$AB = CF = DG$	$CG = DF = AEH$	$ABH = CFH = DGH$
$AC = BF = EGH$	$CH = BDE = AEG$	$ACD = BDF = BCG = AFG$
$AD = BG = EFH$		

2 blocks of 16: $ABE = CEF = DEG$ 4 blocks of 8: $ABE = CEF = DEG$
$ABH = CFH = DGH$
$EH = BCD = ADF = ACG = BFG$

Example 8.6

The 2^{8-3} Design in Four Blocks for Example 8.6

Run	Basic Design A	B	C	D	E	$F = ABC$	$G = ABD$	$H = BCDE$	Block	Actual Run Order	Standard Deviation ($\times 10^3$ in)
1	−	−	−	−	−	−	−	+	3	18	2.76
2	+	−	−	−	−	+	+	+	2	16	6.18
3	−	+	−	−	−	+	+	−	4	29	2.43
4	+	+	−	−	−	−	−	−	1	4	4.01
5	−	−	+	−	−	+	−	−	1	6	2.48
6	+	−	+	−	−	−	+	−	4	26	5.91
7	−	+	+	−	−	−	+	+	2	14	2.39
8	+	+	+	−	−	+	−	+	3	22	3.35
9	−	−	−	+	−	−	+	−	1	8	4.40
10	+	−	−	+	−	+	−	−	4	32	4.10
11	−	+	−	+	−	+	−	+	2	15	3.22
12	+	+	−	+	−	−	+	+	3	19	3.78
13	−	−	+	+	−	+	+	+	3	24	5.32
14	+	−	+	+	−	−	−	+	2	11	3.87
15	−	+	+	+	−	−	−	−	4	27	3.03
16	+	+	+	+	−	+	+	−	1	3	2.95

Example 8.6

17	−	−	−	−	+	−	−	−	2	10	2.64
18	+	−	−	−	+	+	+	−	3	21	5.50
19	−	+	−	−	+	+	+	+	1	7	2.24
20	+	+	−	−	+	−	−	+	4	28	4.28
21	−	−	+	−	+	+	−	+	4	30	2.57
22	+	−	+	−	+	−	+	+	1	2	5.37
23	−	+	+	−	+	−	+	−	3	17	2.11
24	+	+	+	−	+	+	−	−	2	13	4.18
25	−	−	−	+	+	−	+	+	4	25	3.96
26	+	−	−	+	+	+	−	+	1	1	3.27
27	−	+	−	+	+	+	−	−	3	23	3.41
28	+	+	−	+	+	−	+	−	2	12	4.30
29	−	−	+	+	+	+	+	−	2	9	4.44
30	+	−	+	+	+	−	−	−	3	20	3.65
31	−	+	+	+	+	−	−	+	1	5	4.41
32	+	+	+	+	+	+	+	+	4	31	3.40

Example 8.6

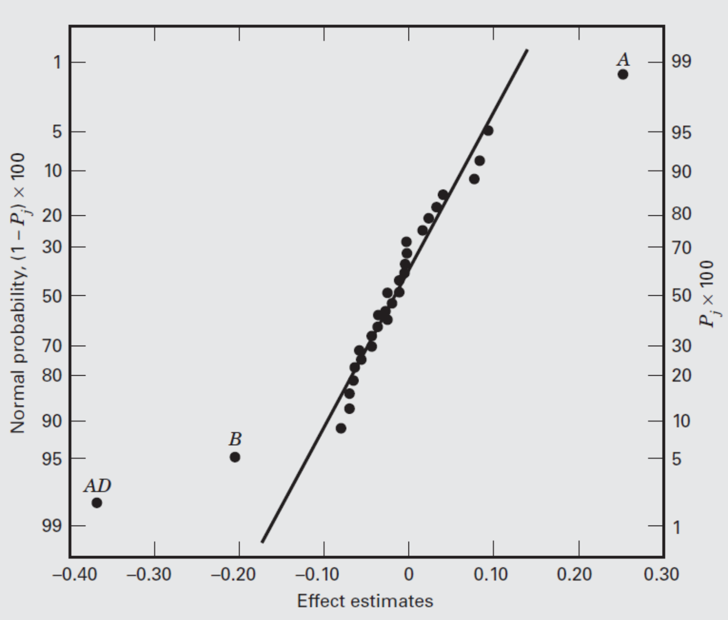
Because the response variable is a standard deviation, it is often best to perform the analysis following a log transformation.

Variable	Regression Coefficient	Estimated Effect	Sum of Squares
Overall average	1.28007		
A	0.14513	0.29026	0.674020
B	−0.10027	−0.20054	0.321729
C	−0.01288	−0.02576	0.005310
D	0.05407	0.10813	0.093540
E	−2.531E-04	−5.063E-04	2.050E-06
F	−0.01936	−0.03871	0.011988
G	0.05804	0.11608	0.107799
H	0.00708	0.01417	0.001606
$AB + CF + DG$	−0.00294	−0.00588	2.767E-04
$AC + BF$	−0.03103	−0.06206	0.030815
$AD + BG$	−0.18706	−0.37412	1.119705
AE	0.00402	0.00804	5.170E-04
$AF + BC$	−0.02251	−0.04502	0.016214
$AG + BD$	0.02644	0.05288	0.022370

Example 8.6

Variable	Regression Coefficient	Estimated Effect	Sum of Squares
AH	−0.02521	−0.05042	0.020339
BE	0.04925	0.09851	0.077627
BH	0.00654	0.01309	0.001371
$CD + FG$	0.01726	0.03452	0.009535
CE	0.01991	0.03982	0.012685
$CG + DF$	−0.00733	−0.01467	0.001721
CH	0.03040	0.06080	0.029568
DE	0.00854	0.01708	0.002334
DH	0.00784	0.01569	0.001969
EF	−0.00904	−0.01808	0.002616
EG	−0.02685	−0.05371	0.023078
EH	−0.01767	−0.03534	0.009993
FH	−0.01404	−0.02808	0.006308
GH	0.00245	0.00489	1.914E-04
ABE	0.01665	0.03331	0.008874
ABH	−0.00631	−0.01261	0.001273
ACD	−0.02717	−0.05433	0.023617

Example 8.6



Normal probability, (1 − P_j) × 100
P_j × 100
Effect estimates
1
5
10
20
30
50
70
80
90
95
99
99
95
90
80
70
50
30
20
10
5
1
–0.40
–0.30
–0.20
–0.10
0
0.10
0.20
0.30
A
B
AD

Example 8.6

Suppose that process knowledge suggests that the appropriate interaction is likely to be AD.

Following table shows the resulting analysis of variance for the model with factors A, B, D, and AD (factor D was included to preserve the hierarchy principle).

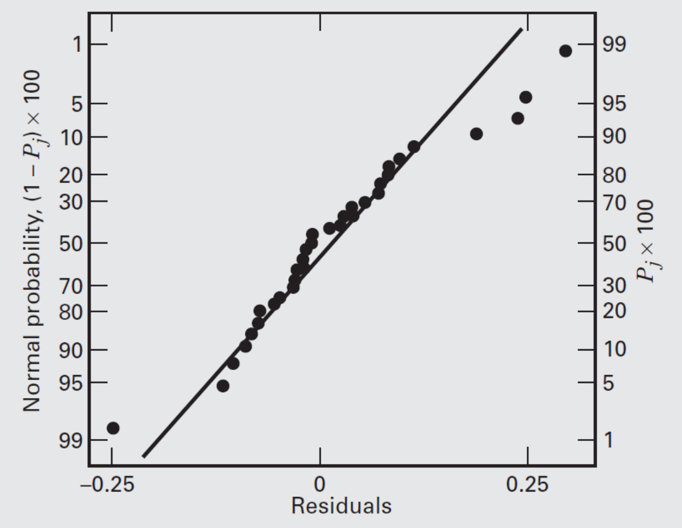
Notice that the block effect is small, suggesting that the machine spindles are not very different.

Example 8.6

Analysis of Variance for Example 8.6

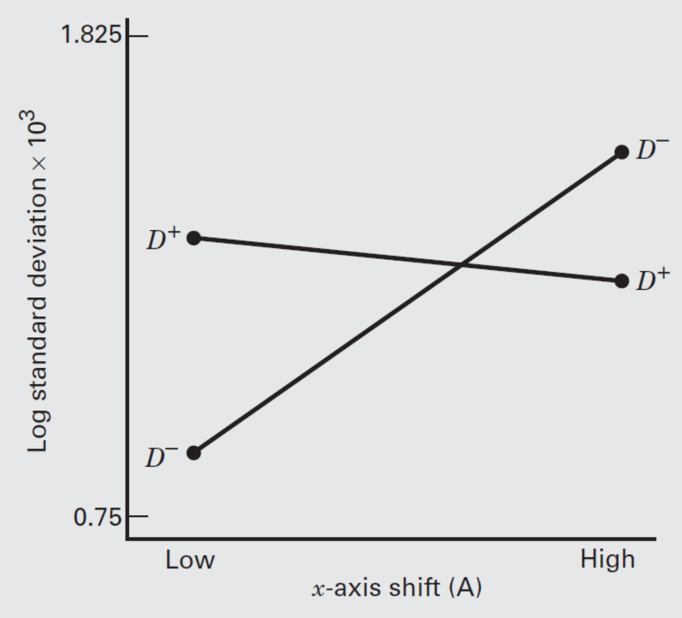
Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
A	0.6740	1	0.6740	39.42	<0.0001
B	0.3217	1	0.3217	18.81	0.0002
D	0.0935	1	0.0935	5.47	0.0280
AD	1.1197	1	1.1197	65.48	<0.0001
Blocks	0.0201	3	0.0067		
Error	0.4099	24	0.0171		
Total	2.6389	31			

Example 8.6



Normal probability, $(1 - P_j) \times 100$
1
5
10
20
30
50
70
80
90
95
99
99
95
90
80
70
50
30
20
10
5
1
$P_j \times 100$
−0.25
0
0.25
Residuals

Example 8.6



1.825
0.75
Log standard deviation × 10^3
D^-
D^+
D^+
D^-
Low
High
x-axis shift (A)

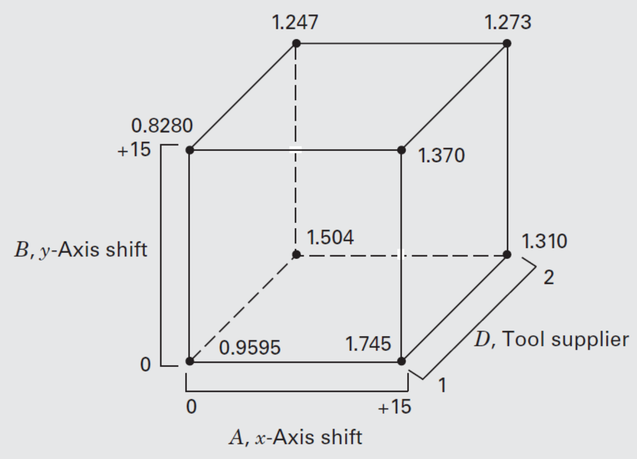
Example 8.6

Normal probability plot of the residuals is suggestive of slightly heavier than normal tails, so possibly other transformations should be considered.

The AD interaction plot shows that running A at the low level (0 offset) and buying tools from supplier 1 gives the best results.

The projection of this design into four replicates of a 2^3 design in factors A, B, and D is shown below.

Example 8.6



The figure indicates that the best combination of operating conditions is A at the low level (0 offset), B at the high level (0.015 in offset), and D at the low level (tool supplier 1).

Exercise 8.45

Consider the design:

Run	A	B	C	D	E	y
1	−1	−1	−1	−1	−1	63
2	1	−1	−1	−1	1	21
3	−1	1	−1	−1	1	36
4	1	1	−1	−1	−1	99
5	−1	−1	1	−1	1	24
6	1	−1	1	−1	−1	66
7	−1	1	1	−1	−1	71
8	1	1	1	−1	1	54
9	−1	−1	−1	1	−1	23
10	1	−1	−1	1	1	74
11	−1	1	−1	1	1	80
12	1	1	−1	1	−1	33
13	−1	−1	1	1	1	63
14	1	−1	1	1	−1	21
15	−1	1	1	1	−1	44
16	1	1	1	1	1	96

Exercise 8.45

1. What is the generator for column E?
2. If ABC is confounded with blocks, run 1 above goes in the block —-. Answer either $+$ or $-$.
3. What is the resolution of this design?
4. Find the estimates of the main effects.