

Chapter 9

(AST301) Design and Analysis of Experiments II

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Lecture Outline

- 1 9. Additional Design and Analysis Topics for Factorial and Fractional Factorial Designs
 - 9.1 The 3^k Factorial Design
 - 9.2 Confounding in the 3^k Factorial Design
 - 9.3 Fractions of 3^k factorial design

Section 1

9. Additional Design and Analysis Topics for Factorial and Fractional Factorial Designs

Subsection 1

9.1 The 3^k Factorial Design

9.1 The 3^k Factorial Design

- The three-level design is written as a 3^k factorial design. It means that k factors are considered, each at 3 levels.
- These are (usually) referred to as low, intermediate and high levels. These levels are numerically expressed as 0, 1, and 2.
- One could have considered the digits -1, 0, and +1, but this may be confusing with respect to the 2-level designs since 0 is reserved for center points. Therefore, we will use the 0, 1, 2 scheme.
- A third level for a continuous factor facilitates investigation of a quadratic relationship between the response and each of the factors.

9.1 The 3^k Factorial Design

- Each treatment combination in the 3^k design is denoted by k digits, where the first digit indicates the level of factor A , the second digit indicates the level of factor B , ..., and the k th digit indicates the level of factor K .
- For example, in a 3^2 design, 00 denotes the treatment combination corresponding to A and B both at the low level, and 01 denotes the treatment combination corresponding to A at the low level and B at the intermediate level.

9.1 The 3^k Factorial Design

- In the 3^k system of designs, when the factors are **quantitative**, we often denote the low, intermediate, and high levels by -1 , 0 , and $+1$, respectively. This facilitates fitting a **regression model** relating the response to the factor levels.
- Regression model for analyzing 3^2 factorial design

$$y = \beta_0 + \beta_1x_1 + \beta_2x_2 + \beta_{12}x_1x_2 + \beta_{11}x_1^2 + \beta_{22}x_2^2$$

9.1 The 3^k Factorial Design

- The 3^k design is useful when curvature of the response surface is concerned, however, there are more useful methods for examining curvature of response surface than the 3^k design, which are
 - ▶ response surface design
 - ▶ 2^k design with center points (central composite design)

Subsection 2

The 3^2 Design

The 3^2 Design

The 3^2 design has two factors, each at three levels. There are $3^2 = 9$ treatment combinations and there are 8 degrees of freedom between these treatment combinations:

00 01 02 10 11 12 20 21 22

- Main effect of A : 2 degrees of freedom
- Main effect of B : 2 degrees of freedom
- Interaction AB : 4 degrees of freedom

For n replicates, there are $3^2n - 1$ total degrees of freedom and $3^2(n - 1)$ degrees of freedom for error

The sums of squares for the effects A , B , and AB can be computed by usual methods discussed in Chapter 5.

The 3^2 Design

The two degrees of freedom of each main effect can be represented by a linear and a quadratic component each with a single degrees of freedom. Suppose that both factors A and B are quantitative

The two-factor interaction AB can be partitioned into two ways:

- 1 Dividing AB into four single-degrees-of-freedom components

$$AB_{L \times L}, AB_{L \times Q}, AB_{Q \times L}, AB_{Q \times Q}$$

This can be done by fitting the terms $\beta_{12}x_1x_2$, $\beta_{122}x_1x_2^2$, $\beta_{112}x_1^2x_2$ and $\beta_{1122}x_1^2x_2^2$ respectively.

- 2 The second method is based on the **orthogonal Latin squares**. Two Latin squares are said to be orthogonal if one square is superimposed on the other, produce all possible ordered pairs of symbols exactly once.

Subsection 3

Example 5.5

Example 5.5

The effective life of a cutting tool installed in a numerically controlled machine is thought to be affected by the cutting speed and the tool angle. Three speeds and three angles are selected, and a 3^2 factorial experiment with two replicates is performed. The coded data are shown below

Data for Tool Life Experiment

Total Angle (degrees)	Cutting Speed (in/min)						$y_{i.}$
	125		150		175		
15	-2		-3		2		-1
	-1	(-3)	0	(-3)	3	(5)	
20	0	(2)	1	(4)	4	(10)	16
	2		3		6		
25	-1	(-1)	5	(11)	0	(-1)	9
	0		6		-1		
$y_{.j}$	-2		12		14		24 = $y_{...}$

Example 5.5

Since the factors are quantitative, and both factors have three levels, a second-order model such as

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{122} x_1 x_2^2 + \beta_{112} x_1^2 x_2 + \beta_{1122} x_1^2 x_2^2$$

can be fit to the data.

From the data we obtain sum of squares as

$$AB_{L \times L} = 8, AB_{L \times Q} = 42.67, AB_{Q \times L} = 2.67 \text{ and } AB_{Q \times Q} = 8$$

That is $SS_{AB} = 8 + 42.67 + 2.67 + 8 = 61.34$.

Example 5.5

The second method is based on **orthogonal Latin squares**. This method does not require that the factors be quantitative, and it is usually associated with the case where all factors are qualitative.

The two factors A and B correspond to the rows and columns, respectively, of a 3×3 Latin square.

For the above example, two particular 3×3 Latin squares are shown superimposed on the cell totals below:

Example 5.5

Treatment combination totals from Example 5.5 with two orthogonal Latin squares superimposed

		Factor B		
		0	1	2
Factor A	0	Q (-3)	R (-3)	S (5)
	1	R (2)	S (4)	Q (10)
	2	S (-1)	Q (11)	R (-1)

(a)

		Factor B		
		0	1	2
Factor A	0	Q (-3)	R (-3)	S (5)
	1	S (2)	Q (4)	R (10)
	2	R (-1)	S (11)	Q (-1)

(b)

Square (a)

$$Q: x_1 + x_2 = 0 \pmod{3}$$

$$R: x_1 + x_2 = 1 \pmod{3}$$

$$S: x_1 + x_2 = 2 \pmod{3}$$

Square (b)

$$Q: x_1 + 2x_2 = 0 \pmod{3}$$

$$S: x_1 = 2x_2 = 1 \pmod{3}$$

$$R: x_1 + 2x_2 = 2 \pmod{3}$$

So, SS from first table is known as AB component of the interaction, and SS from the second table is known as AB^2 component of the interaction

Example 5.5

Total of the letters from the *first table*: $Q = 18$, $R = -2$, $S = 8$. Sum of squares between these totals is 33.3 with 2 df.

Total of the letters from the *second table*: $Q = 0$, $R = 6$, $S = 18$. Sum of squares between these totals is 28 with 2 df.

$$SS_{AB} = 33.3 + 28 = 61.3 \text{ with 4 df}$$

It can also be shown that $AB^2 = (AB^2)^2 = A^2B^4 = A^2B$

Example 5.5

There is another way of calculating the AB and AB^2 components of the interaction:

Add the data by diagonals downward from the **left to right** and the totals are

$$\left. \begin{array}{rcl} -3 + 4 - 1 & = & 0 \\ -3 + 10 - 1 & = & 6 \\ 5 + 2 + 11 & = & 18 \end{array} \right\} \Rightarrow \text{sum of squares of these totals} = 28 = AB^2$$

Add the data by diagonals downward from the **right to left** and the totals are

$$\left. \begin{array}{rcl} 5 + 4 - 1 & = & 8 \\ -3 + 2 - 1 & = & -2 \\ -3 + 11 + 10 & = & 18 \end{array} \right\} \Rightarrow \text{sum of squares of these totals} = 33.3 = AB$$

Example 5.5

Yates called these components of interaction as the *I* and *J* components of interaction:

$$I(AB) = AB^2$$

$$J(AB) = AB$$

Subsection 4

The 3^3 design

The 3^3 design

The 3^3 design has three factors (A, B, C), each has three levels and there $3^3 = 27$ treatment combinations

Among 26 degrees of freedom, 6 for the main effects, 12 for the two-factor interactions, and 8 for the three-factor interaction

For n replicates, there are $3^3n - 1$ total degrees of freedom and $3^3(n - 1)$ degrees of freedom for error

The 3^3 design

Sums of squares can be calculated using the standard methods, in addition to that, if the factors are quantitative

- Main effects can be partitioned into linear (L) and quadratic (Q) components, each with one degrees of freedom
- Two-factor interactions can be partitioned into components corresponding to $L \times L$, $L \times Q$, $Q \times L$, $Q \times Q$
- Three-factor interaction can be partitioned into components corresponding to $L \times L \times L$, $L \times L \times Q$, etc.

The 3^3 design

The sum of squares corresponding to the two-factor interaction has I and J components, e.g. $I(AB) = AB^2$ and $J(AB) = AB$

The sum of squares corresponding to a three-factor interaction has W , X , Y , and Z components, e.g.

$$\begin{aligned}W(ABC) &= AB^2C^2, & X(ABC) &= ABC^2, \\Y(ABC) &= AB^2C, & Z(ABC) &= ABC\end{aligned}$$

Subsection 5

The general 3^k design

The general 3^k design

The 3^k factorial design has k factors, each has 3 levels, so there are 3^k treatment combinations with $3^k - 1$ degrees of freedom between them

There are

- k main effects, each has 2 degrees of freedom
- $\binom{k}{2}$ two-factor interactions, each has $2^2 = 4$ degrees of freedom
- one k -factor interaction, which has 2^k degrees of freedom

If there are n replications then total degrees of freedom will be $n3^k - 1$ and error degrees of freedom is $3^k(n - 1)$

Sums of squares for different effects are computed by usual method of factorial design

The general 3^k design

Any h -factor interaction has 2^{h-1} orthogonal components, each of which has 2 degrees of freedom

E.g. The four-factor interaction $ABCD$ has $2^{4-1} = 8$ orthogonal components, which are

$$\begin{array}{cccc} ABCD^2 & ABC^2D & AB^2CD & ABCD \\ ABC^2D^2 & AB^2CD^2 & AB^2C^2D & AB^2C^2D^2 \end{array}$$

Note that only exponent allowed for the first letter is 1 and if the exponent of the first letter is not 1 then the entire expression must be squared, e.g.

$$A^2BCD = (A^2BCD)^2 = A^4B^2C^2D^2 = AB^2C^2D^2$$

These interaction components have no physical meaning but they are useful in constructing more complex designs

Subsection 6

9.2 Confounding in the 3^k Factorial Design

9.2 Confounding in the 3^k Factorial Design

Even when a single replicate of the 3^k design is considered, the design requires so many runs that it is unlikely that all 3^k runs can be made under uniform conditions.

Thus, confounding in blocks is often necessary. The 3^k design may be confounded in 3^p incomplete blocks, where $p < k$.

Thus, these designs may be confounded in three blocks, nine blocks, and so on.

The 3^k factorial design in three blocks

The three blocks have two degrees of freedom, so there must be *two degrees of freedom* confounded with blocks

In the 3^k factorial design

- each main effect has *two degrees of freedom*
- each two-factor interaction has four degrees of freedom, which can be decomposed into two components (e.g. AB and AB^2), each of which has *two degrees of freedom*
- each three-factor interaction has eight degrees of freedom, which can be decomposed into four components (e.g. ABC , ABC^2 , AB^2C , and AB^2C^2), each of which has *two degrees of freedom*
- and so on

So it is convenient to confound an interaction component with blocks

The 3^k factorial design in three blocks

The general procedure to construct a defining contrast

$$L = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k,$$

where

- α_i represents the exponent on the i^{th} factor in the effect to be confounded
- x_i is the level of the i^{th} factor in a particular treatment combination, x_i can take values either 0 or 1 or 2
- For 3^k design, $\alpha_i = 0, 1, \text{ or } 2$ with first nonzero α_i is unity

L can take on only the values 0, 1, or 2 (mod 3) and the treatment combinations satisfying $L = 0 \pmod{3}$ constitute the principal block, which always contains the treatment combination 00...0

The 3^k factorial design in three blocks

- Suppose goal is to construct a 3^2 factorial design in 3 blocks, one of the components of AB interaction (AB or AB^2) needs to be confounded
- If AB^2 is chosen for confounding with blocks, then the corresponding defining contrast

$$L = x_1 + 2x_2$$

- The value of $L \pmod{3}$ of each treatment combination

$$00 : L = 1(0) + 2(0) = 0 \pmod{3} \quad 12 : L = 1(1) + 2(2) = 2 \pmod{3}$$

$$01 : L = 1(0) + 2(1) = 2 \pmod{3} \quad 20 : L = 1(2) + 2(0) = 2 \pmod{3}$$

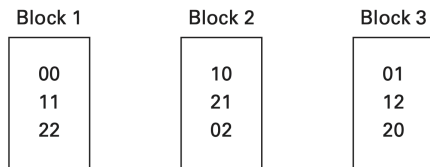
$$02 : L = 1(0) + 2(2) = 1 \pmod{3} \quad 21 : L = 1(2) + 2(1) = 1 \pmod{3}$$

$$10 : L = 1(1) + 2(0) = 1 \pmod{3} \quad 22 : L = 1(2) + 2(2) = 0 \pmod{3}$$

$$11 : L = 1(1) + 2(1) = 0 \pmod{3}$$

The 3^k factorial design in three blocks

Assignment of treatment combinations to blocks



Note that the treatment combinations of principal block form a group with respect to addition (mod 3), i.e. addition of pair of treatment combination will lead to a treatment combination of the principal block. E.g.

$$00 + 11 = 11$$

$$11 + 22 = 00$$

$$22 + 00 = 22$$

The 3^k factorial design in three blocks

Assignment of treatment combinations to blocks

Block 1	Block 2	Block 3
00	10	01
11	21	12
22	02	20

Note that the treatment combinations in the other two blocks may be generated by adding (mod 3) any element of the new block to the elements of principal block. E.g.

$$\text{Block 2 : } 00 + 02 = 02 \quad 11 + 02 = 10 \quad 22 + 02 = 21$$

$$\text{Block 3 : } 00 + 20 = 20 \quad 11 + 20 = 01 \quad 22 + 20 = 12$$

Example 9.2

The statistical analysis of the 3^2 design confounded in three blocks is illustrated by using the following data

	Block 1	Block 2	Block 3									
	<table border="1"><tr><td>00 = 4</td></tr><tr><td>11 = -4</td></tr><tr><td>22 = 0</td></tr></table>	00 = 4	11 = -4	22 = 0	<table border="1"><tr><td>10 = -2</td></tr><tr><td>21 = 1</td></tr><tr><td>02 = 8</td></tr></table>	10 = -2	21 = 1	02 = 8	<table border="1"><tr><td>01 = 5</td></tr><tr><td>12 = -5</td></tr><tr><td>20 = 0</td></tr></table>	01 = 5	12 = -5	20 = 0
00 = 4												
11 = -4												
22 = 0												
10 = -2												
21 = 1												
02 = 8												
01 = 5												
12 = -5												
20 = 0												
Block totals =	0	7	0									

We find that

$$SS_{\text{Blocks}} = \frac{(0)^2 + (7)^2 + (0)^2}{3} - \frac{(7)^2}{9} = 10.89$$

Example 9.2

SS for main effects and interaction:

Factor <i>A</i>	Factor <i>B</i>			Total
	0	1	2	
0	4	5	8	17
1	-2	-4	-5	-11
2	0	1	0	1
Total	2	2	3	7

$$SS_A : \frac{17^2 + 11^2 + 1^2}{3} - \frac{7^2}{9} = 131.56$$

$$SS_B : \frac{2^2 + 2^2 + 3^2}{3} - \frac{7^2}{9} = 0.22$$

$$SS_{AB} : \frac{4^2 + 3^2 + 0^2}{3} - \frac{7^2}{9} = 2.89$$

$$SS_{AB^2} : \frac{0^2 + 0^2 + 7^2}{3} - \frac{7^2}{9} = 10.89$$

Example 9.2

The I or AB^2 component of the AB interaction may be found by computing the sum of squares between the left-to-right diagonal totals in the above layout. SS_{Blocks} is exactly equal to the AB^2 component of interaction.

The analysis of variance is shown below. Because there is only one replicate, no formal tests can be performed. It is not a good idea to use the AB component of interaction as an estimate of error.

■ **TABLE 9.4**

Analysis of Variance for Data in Example 9.2

Source of Variation	Sum of Squares	Degrees of Freedom
Blocks (AB^2)	10.89	2
A	131.56	2
B	0.22	2
AB	2.89	2
Total	145.56	8

Example 9.2

Slightly complicated example: The 3^3 design confounded in three blocks of nine runs each. Suppose the AB^2C^2 component of the three-factor interaction will be confounded with blocks, the defining contrast is

$$L = x_1 + 2x_2 + 2x_3$$

It is clear that the treatment combinations 000, 110 and 101 are elements of principal block, other six elements of principal block can be obtained from these elements:

$$110 + 110 = 220$$

$$101 + 101 = 202$$

$$110 + 101 = 211$$

$$220 + 101 = 021$$

$$202 + 220 = 122$$

$$202 + 110 = 012$$

Example 9.2

Treatment combinations of the principal block of the 3^3 design with AB^2C^2 component confounded in three blocks.

Block 1	Block 2	Block 3
000	001	000+011=011
012	010	012+011=020
021	022	021+011=002
101	102	101+011=112
110	111	110+011=121
122	120	122+011=100
211	212	211+011=222
220	221	220+011=201
202	200	202+011=210

Treatment combinations of other two blocks can easily be obtained by adding a treatment combination (which is not in the principal block) to each element of the principal block

Example 9.2

■ **TABLE 9.5**

Analysis of Variance for a 3^3 Design with AB^2C^2 Confounded

Source of Variation	Degrees of Freedom
Blocks (AB^2C^2)	2
<i>A</i>	2
<i>B</i>	2
<i>C</i>	2
<i>AB</i>	4
<i>AC</i>	4
<i>BC</i>	4
Error ($ABC + AB^2C + ABC^2$)	6
Total	26

The 3^k factorial design in nine blocks

- If 3^k design is confounded in 9 blocks then 8 degrees of freedom is confounded with blocks.
- We need to choose two interaction components (4 degrees of freedom), which will result two more components will be confounded (4 degrees of freedom)
- E.g. if P and Q are two components that are selected for confounding then PQ and PQ^2 will also be confounded with blocks
- There will be two defining contrasts

$$L_1 = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$$

$$L_2 = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k$$

9 blocks can be constructed based on the values of L_1 and L_2 or using group-theoretic property of the principal block

The 3^k factorial design in nine blocks

Consider the 3^4 factorial design confounded in nine blocks of nine runs each.

Suppose we choose to confound ABC and AB^2D^2 . Their generalized interactions

$$(ABC)(AB^2D^2) = A^2B^3CD^2 = (A^2B^3CD^2)^2 = AC^2D$$

$$(ABC)(AB^2D^2)^2 = A^3B^5CD^4 = B^2CD = (B^2CD)^2 = BC^2D^2$$

are also confounded with blocks. The defining contrasts for ABC and AB^2D^2 are

$$L_1 = x_1 + x_2 + x_3$$

$$L_2 = x_1 + 2x_2 + 2x_4$$

The 3^k factorial design in nine blocks

Block 1	Block 2	Block 3	Block 4	Block 5	Block 6	Block 7	Block 8	Block 9
0000	0001	2000	0200	0020	0010	1000	0100	0002
0122	0120	2122	0022	0112	0102	1122	0222	0121
0211	0212	2211	0111	0201	0221	1211	0011	0210
1021	1022	0021	1221	1011	1001	2021	1121	1020
1110	1111	0110	1010	1100	1120	2110	1210	1112
1202	1200	0202	1102	1222	1212	2202	1002	1201
2012	2010	1012	2212	2002	2022	0012	2112	2011
2101	2102	1101	2001	2121	2111	0101	2201	2100
2220	2221	1220	2120	2210	2200	0220	2020	2222

$(L_1, L_2) = (0,0)$

$(0,2)$

$(2,2)$

$(2,1)$

$(2,0)$

$(1,6)$

$(1,1)$

$(1,2)$

$(0,1)$

Subsection 7

9.3 Fractions of 3^k factorial design

The one-third fraction of the 3^k factorial design

- The largest fraction of the 3^k design is a one-third fraction contains 3^{k-1} runs, which is known as 3^{k-1} fractional design
- To construct a 3^{k-1} fractional factorial design, select an interaction component (which has 2 degrees of freedom) and partition the full 3^k runs into three blocks.
- Each of the three resulting blocks is a 3^{k-1} fractional design, and any one of the blocks may be selected for use
- If $AB^{\alpha_2}C^{\alpha_3} \dots K^{\alpha_k}$ is the component of interaction used to define the blocks, then the defining relation is

$$I = AB^{\alpha_2}C^{\alpha_3} \dots K^{\alpha_k}$$

- The alias structure for an effect can be obtained by multiplying the effect by both I and I^2 , e.g. alias structure of the main effect A are AI and AI^2 .

The one-third fraction of the 3^k factorial design

- **Example:** To construct a one-third fraction of the 3^3 design, we may consider any of the interaction components as defining relation.
- Since there are four interaction components, which are ABC , ABC^2 , AB^2C , AB^2C^2 , there will be 12 possible one-third fraction of the 3^3 design and the defining contrast to obtain the designs

$$x_1 + \alpha_2 x_2 + \alpha_3 x_3 = u \pmod{3}$$

where $\alpha = 1, 2$ and $u = 0, 1, \text{ or } 2$.

- If we select the component AB^2C^2 , the 3^{3-1} design will contain exactly 9 treatment combinations that must satisfy

$$x_1 + 2x_2 + 2x_3 = u, \pmod{3}$$

where $u = 0, 1, \text{ or } 2$.

The one-third fraction of the 3^k factorial design

The alias structure with the defining relation $I = AB^2C^2$:

$$A = A(AB^2C^2) = A^2B^2C^2 = ABC$$

$$A = A(AB^2C^2)^2 = A^3B^4C^4 = BC$$

$$B = B(AB^2C^2) = AB^3C^2 = AC^2$$

$$B = B(AB^2C^2)^2 = A^2B^5C^4 = ABC^2$$

$$C = C(AB^2C^2) = AB^2C^3 = AB^2$$

$$C = C(AB^2C^2)^2 = A^2B^4C^5 = AB^2C$$

$$AB = AB(AB^2C^2) = A^2B^3C^2 = AC$$

$$AB = AB(AB^2C^2)^2 = A^3B^5C^4 = BC^2$$

The four effects that are actually estimated from the eight degrees of freedom are: $A + BC + ABC$, $B + BC^2 + ABC^2$, $C + AB^2 + AB^2C$, $AB + AC + BC^2$

The one-third fraction of the 3^k factorial design

The treatment combinations in a 3^{k-1} design with the defining relation

$$I = AB^{\alpha_2}C^{\alpha_3} \dots K^{\alpha_k}$$

can be constructed as:

- Write down the 3^{k-1} runs for a full factorial design with $k - 1$ factors, with the usual 0, 1, 2 notation.
- The k^{th} factor is introduced by equating its levels x_k to the appropriate component of the highest order interaction $AB^{\alpha_2} \dots (K - 1)^{\alpha_{k-1}}$ through the relationship

$$x_k = \beta_1 x_1 + \dots + \beta_{k-1} x_{k-1},$$

where $\beta_i = (3 - \alpha_k)\alpha_i \pmod{3}$ for $1 \leq i \leq k - 1$. This yields a design of the highest possible resolution

The one-third fraction of the 3^k factorial design

Consider a one-third fraction of the 3^4 factorial design with $I = AB^2CD$, i.e. $x_1 + 2x_2 + x_3 + x_4 = u \pmod{3}$.

First construct the 3^3 design in factor A , B , and C :

Design 1 $u = 0$	Design 1 $u = 1$	Design 1 $u = 2$
000	100	200
012	112	212
101	201	001
202	002	102
021	121	221
110	210	010
122	222	022
211	011	111
220	020	120

Levels for the factor D can be obtained from

$$x_4 = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3, \quad \beta_i = (3 - \alpha_4) \alpha_i, \quad i = 1, 2, 3$$

The one-third fraction of the 3^k factorial design

Defining relation: $x_1 + 2x_2 + x_3 + x_4 = u \pmod{3}$

$\Rightarrow \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 1, \alpha_4 = 1$ Levels for factor D:

$x_4 = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$, where $\beta_1 = (3 - \alpha_4)\alpha_1 = 2$, $\beta_2 = 1$, and $\beta_3 = 2$

$\Rightarrow x_4 = 2x_1 + x_2 + 2x_3$

■ TABLE 9.6

A 3_{IV}^{4-1} Design with $I = AB^2CD$

0000	0012	2221
0101	0110	0021
1100	0211	0122
1002	1011	0220
0202	1112	1020
1201	1210	1121
2001	2010	1222
2102	2111	2022
2200	2212	2120

As an example, alias structure for the effect A is

$$A(AB^2CD) = A^2B^2CD = ABC^2D^2, A(AB^2CD)^2 = BC^2D^2$$

The general 3^{k-p} fractional factorial design

- The 3^{k-p} fractional factorial design is th $(1/3)^p$ fraction of the 3^k design for $p < k$, where the fraction contains 3^{k-p} runs.
- The construction of 3^{k-p} includes the selection of p components of interaction first, which is then used to partition 3^k treatment combinations into 3^p blocks. Each block is a 3^{k-p} fractional factorial design.
- The defining relation I of any fraction consists of p initially chosen components and their $(3^p - 2p - 1)/2$ generalized interactions.
- Alias structure of an effect can be obtained by multiplying the effect by the defining relation I and I^2
- The 3^{k-p} fractional factorial design can also be obtained by writing down the full 3^{k-p} design first in $(k - p)$ factors and then levels of additional p factors are obtained from the corresponding defining contrasts.