Chapter 13

(AST301) Design and Analysis of Experiments II

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Lecture Outline

1. 13. Experiments with Random Factors
   - 13.1 Introduction
   - 13.2 The Two-Factor Factorial with Random Factors
   - 13.3 The Two-Factor Mixed Model
   - 13.4 Rules for Expected Mean Squares
   - 13.5 Approximate F-Tests

Section 1

13. Experiments with Random Factors

Subsection 1

13.1 Introduction

13.1 Introduction

Throughout most of this book we have assumed that the factors in an experiment were **fixed factors**, that is, the levels of the factors used by the experimenter were the specific levels of interest.

The implication of this, of course, is that the statistical inferences made about these factors are confined to the specific levels studied.

That is, if three material types are investigated as in the battery life experiment of Example 5.1, our conclusions are valid only about those specific material types.

13.1 Introduction

A variation of this occurs when the factor or factors are **quantitative**. In these situations, we often use a regression model relating the response to the factors to predict the response over the region spanned by the factor levels used in the experimental design.

Several examples of this were presented in Chapters 5 through 9. In general, with a fixed effect, we say that the **inference space** of the experiment is the specific set of factor levels investigated.

13.1 Introduction

In some experimental situations, the factor levels are chosen at **random** from a larger population of possible levels, and the experimenter wishes to draw conclusions about the entire population of levels, not just those that were used in the experimental design.

In this situation, the factor is said to be a **random factor**.

The random effect model was introduced in Chapter 3 for a single-factor experiment, and we used that to introduce the **random effects model** for the analysis of variance and **components of variance**.

13.1 Introduction

For example: a company has 50 machines that make cardboard cartons for canned goods, and they want to understand the variation in strength of the cartons.

They choose ten machines at random from the 50 and make 40 cartons on each machine, assigning 400 lots of feedstock cardboard at random to the ten chosen machines.

The resulting cartons are tested for strength. This is a completely randomized design, with ten treatments and 400 units.

13.1 Introduction

Fixed Effects Model

$$y_{ij} = \mu + \tau_i + \epsilon_{ij}$$
$$\epsilon_{ij} \sim N(0, \sigma^2)$$

- μ: overall mean
- τ_i: fixed effect of treatment i
- ϵ_{ij}: random error
- τ_i are **fixed unknown parameters**

Random Effects Model

$$y_{ij} = \mu + \tau_i + \epsilon_{ij}$$
$$\tau_i \sim N(0, \sigma_\tau^2)$$
$$\epsilon_{ij} \sim N(0, \sigma^2)$$

- μ: overall mean
- τ_i: random effect of treatment i
- ϵ_{ij}: random error
- τ_i are **random variables**

- Notice that we still decompose the model into: Overall mean (μ), Treatment effect (τ_i), Random error (ϵ_{ij})
- **Why Fixed-Effects Assumptions Don't Make Sense in Random Effects Model?**

13.1 Introduction

1. Treatment levels are not fixed but randomly sampled

- In the fixed-effects model, the treatment levels (e.g., different brands, machines, or methods) are **specifically chosen and of interest**.
- In the random-effects model, these levels are assumed to be a **random sample from a larger population** of possible treatments.
- Therefore, estimating individual treatment effects (τ_i) is less meaningful — we care more about the **variation among treatments**, not their specific values.

13.1 Introduction

2. The focus shifts from estimation to generalization

- In fixed-effects, we want to **compare specific treatment effects**.
- In random-effects, we aim to **generalize** to the broader population of treatments.
- So, we're more interested in estimating **variance components** (like σ_τ^2) to understand how much treatments vary, not just how they differ.

13.1 Introduction

3. Inference is about variance components

- In random-effects, variability in treatment levels is treated as **another source of random variation**.
- This affects how we **partition the total variance** and how we perform **statistical inference** (like testing and confidence intervals).

13.1 Introduction

In this chapter, we focus on methods for the design and analysis of factorial experiments with random factors.

In Chapter 14, we will present **nested** and **split-plot designs**, two situations where random factors are frequently encountered in practice.

Review: Random Effects Model

- Random effects model is defined only for the random factors, e.g.

$$y_{ij} = \mu + \tau_i + \epsilon_{ij}, \quad i = 1, \dots, a;\ j = 1, \dots, n$$

  where both τ_i and ϵ_{ij} are random variables (τ_i is not parameter), which are assumed to follow $\mathcal{N}(0, \sigma_\tau^2)$ and $\mathcal{N}(0, \sigma^2)$, respectively.

- τ_i and ϵ_{ij} are independent
- Variance structure

$$\text{cov}(y_{ij}, y_{i'j'}) = \begin{cases} \sigma_\tau^2 + \sigma^2 & \text{if} \quad i = i',\ j = j' \\ \sigma_\tau^2 & \text{if} \quad i = i',\ j \neq j \\ 0 & \text{if} \quad i \neq i' \end{cases}$$

- σ_τ^2 and σ^2 are known as variance components

Review: Random Effects Model

The parameters of the random effects model are the overall mean μ, the error variance σ^2, and the variance of the treatment effects σ_τ^2; the treatment effects τ_i are random variables, not parameters.

We want to make inferences about these parameters; we are not so interested in making inferences about the τ_i 's and ϵ_{ij} 's.

Typical inferences would be point estimates or confidence intervals for the variance components, or a test of the null hypothesis that the treatment variance σ_τ^2 is 0

Review: Random Effects Model

- Hypothesis considered for the fixed effects model

$$H_0 : \text{no difference between treatment levels}$$

  is no longer useful for the random effects model

- For random effects model the hypothesis regarding no treatment effects is defined as

$$H_0 : \sigma^2_\tau = 0 \quad vs \quad H_1 : \sigma^2_\tau > 0$$

Review: Random Effects Model

- For random effects model, the sum of squares identity

$$SS_T = SS_{Treat} + SS_E$$

  remains valid

- It can be shown

$$E(MS_{Treat}) = \sigma^2 + n\sigma_\tau^2 \quad \text{and} \quad E(MS_E) = \sigma^2$$

- Under the null hypothesis $H_0 : \sigma_\tau^2 = 0$, the statistic

$$F_0 = \frac{MS_{Treat}}{MS_E}$$

  follows a F-distribution with $(a-1)$ and $a(n-1)$ degrees of freedom

Review: Random Effects Model

- Beside hypothesis testing, estimation of random effects parameters is also of interest in analyzing random effects models
- We have

$$E(MS_{Treat}) = \sigma^2 + n\sigma_\tau^2 \quad \text{and} \quad E(MS_E) = \sigma^2$$

so the unbiased estimators of σ^2 and σ_τ^2 are

$$\hat{\sigma}^2 = MS_E \quad \text{and} \quad \hat{\sigma}_\tau^2 = \frac{MS_{Treat} - MS_E}{n}$$

Review: Random Effects Model

- CI of σ^2 can be constructed using the result

$$\frac{a(n-1)MS_E}{\sigma^2} \sim \chi^2_{a(n-1)}$$

- We can write the $100(1-\alpha)\%$ CI

$$\text{pr}\left[\chi^2_{a(n-1),\alpha/2} \leq \frac{a(n-1)MS_E}{\sigma^2} \leq \chi^2_{a(n-1),1-\alpha/2}\right] = 1-\alpha$$

$$\text{pr}\left[\frac{a(n-1)MS_E}{\chi^2_{a(n-1),1-\alpha/2}} \leq \sigma^2 \leq \frac{a(n-1)MS_E}{\chi^2_{a(n-1),\alpha/2}}\right] = 1-\alpha$$

Review: Random Effects Model

- The CI for σ_τ^2 is not straight forward, but it is easy to obtain the CI for $\sigma_\tau^2/(\sigma^2 + \sigma_\tau^2)$ and σ_τ^2/σ^2 using the result

$$\frac{MS_{Treat}/(n\sigma_\tau^2 + \sigma^2)}{MS_E/\sigma^2} \sim F_{a-1,a(n-1)}$$

Example 3.11

A textile company weaves a fabric on a large number of looms. It would like the looms to be homogeneous so that it obtains a fabric of uniform strength. The process engineer suspects that, in addition to the usual variation in strength within samples of fabric from the same loom, there may also be significant variations in strength between looms. To investigate this, she selects four looms at random and makes four strength determinations on the fabric manufactured on each loom. This experiment is run in random order, and the data obtained are shown in Table 3.17.

Example 3.11

■ **TABLE 3.17**

Strength Data for Example 3.11

Looms	**Observations** 1	2	3	4	$y_{i\cdot}$
1	98	97	99	96	390
2	91	90	93	92	366
3	96	95	97	95	383
4	95	96	99	98	388
					$1527 = y_{\cdot\cdot}$

The standard ANOVA partition of the sum of squares is appropriate. There is nothing new in terms of computing.

■ **TABLE 3.18**

Analysis of Variance for the Strength Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Looms	89.19	3	29.73	15.68	<0.001
Error	22.75	12	1.90		
Total	111.94	15			

Example 3.11

From the ANOVA, we conclude that the looms in the plant differ significantly.

The variance components are estimated by $\hat{\sigma}^2 = 1.90$ and

$$\hat{\sigma}_\tau^2 = \frac{29.73 - 1.90}{4} = 6.96$$

Therefore, the variance of any observation on strength is estimated by

$$\hat{\sigma}_y = \hat{\sigma}^2 + \hat{\sigma}_\tau^2 = 1.90 + 6.96 = 8.86.$$

Most of this variability is attributable to differences between looms.

Subsection 2

13.2 The Two-Factor Factorial with Random Factors

13.2 The Two-Factor Factorial with Random Factors

- Two factors A and B, a levels of A and b levels of B are randomly selected in the experiment. The model

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk},$$

where τ_i, β_j, $(\tau\beta)_{ij}$, and ϵ_{ijk} are random

- Assumptions

$$\tau_i \sim \mathcal{N}(0, \sigma_\tau^2), \quad \beta_j \sim \mathcal{N}(0, \sigma_\beta^2), \quad (\tau\beta)_{ij} \sim \mathcal{N}(0, \sigma_{\tau\beta}^2), \quad \epsilon_{ijk} \sim \mathcal{N}(0, \sigma^2)$$

- $V\left(y_{ijk}\right) = \sigma_\tau^2 + \sigma_\beta^2 + \sigma_{\tau\beta}^2 + \sigma^2$
- Hypotheses of interest

$$(a)\, H_0 : \sigma_\tau^2 = 0 \quad \text{against} \quad H_1 : \sigma_\tau^2 > 0$$
$$(b)\, H_0 : \sigma_\beta^2 = 0 \quad \text{against} \quad H_1 : \sigma_\beta^2 > 0$$
$$(c)\, H_0 : \sigma_{\tau\beta}^2 = 0 \quad \text{against} \quad H_1 : \sigma_{\tau\beta}^2 > 0$$

13.2 The Two-Factor Factorial with Random Factors

- The form of the test statistics depend on the **expected mean squares**
- Expected mean squares

$$
\begin{aligned}
E(MS_A) &= \sigma^2 + n\sigma^2_{\tau\beta} + bn\sigma^2_{\tau} \\
E(MS_B) &= \sigma^2 + n\sigma^2_{\tau\beta} + an\sigma^2_{\beta} \\
E(MS_{AB}) &= \sigma^2 + n\sigma^2_{\tau\beta} \\
E(MS_E) &= \sigma^2
\end{aligned}
$$

- Test statistic for $H_0 : \sigma^2_{\tau\beta} = 0$

$$F_0 = \frac{MS_{AB}}{MS_E} \sim F_{(a-1)(b-1),ab(n-1)}$$

13.2 The Two-Factor Factorial with Random Factors

- Expected mean squares

$$
\begin{aligned}
E(MS_A) &= \sigma^2 + n\sigma^2_{\tau\beta} + bn\sigma^2_{\tau} \\
E(MS_B) &= \sigma^2 + n\sigma^2_{\tau\beta} + an\sigma^2_{\beta} \\
E(MS_{AB}) &= \sigma^2 + n\sigma^2_{\tau\beta} \\
E(MS_E) &= \sigma^2
\end{aligned}
$$

- Test statistic for $H_0 : \sigma^2_{\tau} = 0$

$$
F_0 = \frac{MS_A}{MS_{AB}} \sim F_{(a-1),(a-1)(b-1)}
$$

13.2 The Two-Factor Factorial with Random Factors

- Expected mean squares

$$\begin{aligned} E(MS_A) &= \sigma^2 + n\sigma^2_{\tau\beta} + bn\sigma^2_{\tau} \\ E(MS_B) &= \sigma^2 + n\sigma^2_{\tau\beta} + an\sigma^2_{\beta} \\ E(MS_{AB}) &= \sigma^2 + n\sigma^2_{\tau\beta} \\ E(MS_E) &= \sigma^2 \end{aligned}$$

- Test statistic for $H_0 : \sigma^2_{\beta} = 0$

$$F_0 = \frac{MS_B}{MS_{AB}} \sim F_{(b-1),(a-1)(b-1)}$$

13.2 The Two-Factor Factorial with Random Factors

Notice that these test statistics are not the same as those used if both factors A and B are fixed.

The expected mean squares are always used as a guide to test statistic construction.

In many experiments involving random factors, interest centers at least as much on estimating the variance components as on hypothesis testing.

13.2 The Two-Factor Factorial with Random Factors

- Estimates of the variance components

$$\hat{\sigma}^2 = MS_E$$

$$\hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_E}{n}$$

$$\hat{\sigma}^2_{\tau} = \frac{MS_A - MS_E}{bn}$$

$$\hat{\sigma}^2_{\beta} = \frac{MS_B - MS_E}{an}$$

A Measurement Systems Capability Study

- A Measurement System Capability Study (also called **Gauge R&R study**, where R&R stands for Repeatability and Reproducibility) is a key part of quality control and process improvement — especially in manufacturing and lab settings.
- **Gauge R&R study** evaluates how much variation in your measurement data is coming from:
  - The **actual process or product** you're measuring
  - The **measurement system** itself (which includes the instrument and the operator)
- In short, it tells you: **"Can we trust our measurement system?"**

A Measurement Systems Capability Study

Main Goals

- Assess how **precise** and **reliable** your measurements are
- Quantify measurement error
- Determine whether your measurement system is suitable for use in a process control or quality monitoring environment

A Measurement Systems Capability Study

Two Key Components

1. Repeatability Variation when the **same operator** measures the same item multiple times using the same instrument.
2. Reproducibility Variation between operators (or appraisers), i.e., when **different people** measure the same item using the same instrument.

A Measurement Systems Capability Study

Basic Experimental Setup

To perform a Gauge R&R study, you typically:

- Choose n parts from the process (covering the process range)
- Have m operators
- Each operator measures each part r times (repeated measures)

A Measurement Systems Capability Study

(Example 13.1)

A typical gauge R&R experiment is shown in Table 13.1. An instrument or gauge is used to measure a critical dimension on a part.

Twenty parts have been selected from the production process, and **three randomly selected operators** measure each part **twice** with this gauge.

The order in which the measurements are made is completely randomized, so this is a two-factor factorial experiment with design factors parts and operators, with 2 replications.

Both parts and operators are random factors. So, we're more interested in estimating variance components than testing specific factor levels.

A Measurement Systems Capability Study

■ **TABLE 13.1**

The Measurement Systems Capability Experiment in Example 13.2

Part Number	Operator 1		Operator 2		Operator 3	
1	21	20	20	20	19	21
2	24	23	24	24	23	24
3	20	21	19	21	20	22
4	27	27	28	26	27	28
5	19	18	19	18	18	21
6	23	21	24	21	23	22
7	22	21	22	24	22	20
8	19	17	18	20	19	18
9	24	23	25	23	24	24
10	25	23	26	25	24	25
11	21	20	20	20	21	20
12	18	19	17	19	18	19
13	23	25	25	25	25	25
14	24	24	23	25	24	25
15	29	30	30	28	31	30
16	26	26	25	26	25	27
17	20	20	19	20	20	20
18	19	21	19	19	21	23
19	25	26	25	24	25	25
20	19	19	18	17	19	17

A Measurement Systems Capability Study

Let:

$$y_{ijk} = \mu + P_i + O_j + (PO)_{ij} + \epsilon_{ijk}$$

Where:

- y_{ijk}: the k-th measurement of part i by operator j
- μ: overall mean
- $P_i \sim N(0, \sigma_P^2)$: random effect of the i-th part
- $O_j \sim N(0, \sigma_O^2)$: random effect of the j-th operator
- $(PO)_{ij} \sim N(0, \sigma_{PO}^2)$: interaction between part and operator
- $\epsilon_{ijk} \sim N(0, \sigma^2)$: repeatability (pure measurement error)

A Measurement Systems Capability Study

■ **TABLE 13.2**

Analysis of Variance (Minitab Balanced ANOVA) for Example 13.1

```
Analysis of Variance (Balanced Designs)

Factor     Type  Levels  Values
part     random     20     1     2     3     4     5     6     7
                           8     9    10    11    12    13    14
                          15    16    17    18    19    20
operator random    3       1     2     3

Analysis of Variance for  y

Source          DF         SS        MS       F        P
part            19   1185.425    62.391   87.65    0.000
operator         2      2.617     1.308    1.84    0.173
part*operator   38     27.050     0.712    0.72    0.861
Error           60     59.500     0.992
Total          119   1274.592
```

A Measurement Systems Capability Study

Estimating Variance Components

Using **Method of Moments** we can estimate:

$$\hat{\sigma}^2 \quad \text{(repeatability)} = 0.99$$

$$\hat{\sigma}^2_P \quad \text{(part variation)} = \frac{62.39 - 0.71}{(3)(2)} = 10.28$$

$$\hat{\sigma}^2_O \quad \text{(operator variation)} = \frac{1.31 - 0.71}{(20)(2)} = 0.015$$

$$\hat{\sigma}^2_{PO} \quad \text{(part-operator interaction)} = \frac{0.71 - 0.99}{2} = -0.14$$

A Measurement Systems Capability Study

As interaction is not significant, the reduced model is
$y_{ijk} = \mu + P_i + O_j + \epsilon_{ijk}$

■ **TABLE 13.3**
Analysis of Variance for the Reduced Model, Example 13.1

Analysis of Variance (Balanced Designs)

Factor	Type	Levels	Values						
part	random	20	1	2	3	4	5	6	7
			8	9	10	11	12	13	14
			15	16	17	18	19	20	
operator	random	3	1	2	3				

Analysis of Variance for y

Source	DF	SS	MS	F	P
part	19	1185.425	62.391	70.64	0.000
operator	2	2.617	1.308	1.48	0.232
Error	98	86.550	0.883		
Total	119	1274.592			

A Measurement Systems Capability Study

$$\hat{\sigma}_P^2 = \frac{62.39 - 0.88}{(3)(2)} = 10.25$$

$$\hat{\sigma}_O^2 \quad \text{(reproducibility)} = \frac{1.31 - 0.88}{(20)(2)} = 0.0108$$

$$\hat{\sigma}^2 \quad \text{(repeatability)} = 0.88$$

A Measurement Systems Capability Study

Finally, we could estimate the **variance of the gauge** as the sum of the variance component estimates $\hat{\sigma}^2$ and $\hat{\sigma}_O^2$ as

$$\begin{aligned}\hat{\sigma}^2_{\text{gauge}} &= \hat{\sigma}^2 + \hat{\sigma}_O^2 \\ &= 0.88 + 0.0108 \\ &= 0.8908\end{aligned}$$

The variability in the gauge appears small relative to the variability in the product.

This is generally a desirable situation, implying that the gauge is capable of distinguishing among different grades of product.

Subsection 3

13.3 The Two-Factor Mixed Model

13.3 The Two-Factor Mixed Model

- Suppose the levels of the factor A are fixed and the levels of factor B are random
- The two-factor mixed model can be expressed as

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk},$$

  where τ_i is fixed, and β_j, $(\tau\beta)_{ij}$ and ϵ_{ij} are random
- Assumptions

$$\beta_j \sim \mathcal{N}(0, \sigma^2_\beta), \quad \epsilon_{ij} \sim \mathcal{N}(0, \sigma^2), \quad (\tau\beta)_{ij} \sim \mathcal{N}(0, \sigma^2_{\tau\beta}(a-1)/a)$$

- Restrictions: $\sum_i \tau_i = 0$, $\sum_i (\tau\beta)_{ij} = 0$
- This type of mixed model is known as **restricted mixed model**

13.3 The Two-Factor Mixed Model

- The expected value of the mean squares

$$\begin{aligned} E(MS_A) &= \sigma^2 + n\sigma^2_{\tau\beta} + \frac{bn\sum_i \tau_i^2}{a-1} \\ E(MS_B) &= \sigma^2 + an\sigma^2_{\beta} \\ E(MS_{AB}) &= \sigma^2 + n\sigma^2_{\tau\beta} \\ E(MS_E) &= \sigma^2 \end{aligned}$$

- Test statistic for $H_0 : \tau_i = 0, \quad \forall \quad i$

$$F_0 = \frac{MS_A}{MS_{AB}} \sim F_{a-1,(a-1)(b-1)}$$

13.3 The Two-Factor Mixed Model

- The expected value of the mean squares

$$
\begin{aligned}
E(MS_A) &= \sigma^2 + n\sigma^2_{\tau\beta} + \frac{bn\sum_i \tau_i^2}{a-1} \\
E(MS_B) &= \sigma^2 + an\sigma^2_{\beta} \\
E(MS_{AB}) &= \sigma^2 + n\sigma^2_{\tau\beta} \\
E(MS_E) &= \sigma^2
\end{aligned}
$$

- Test statistic for $H_0 : \sigma^2_\beta = 0$

$$F_0 = \frac{MS_B}{MS_E} \sim F_{b-1,ab(n-1)}$$

13.3 The Two-Factor Mixed Model

- The expected value of the mean squares

$$E(MS_A) = \sigma^2 + n\sigma^2_{\tau\beta} + \frac{bn\sum_i \tau_i^2}{a-1}$$
$$E(MS_B) = \sigma^2 + an\sigma^2_{\beta}$$
$$E(MS_{AB}) = \sigma^2 + n\sigma^2_{\tau\beta}$$
$$E(MS_E) = \sigma^2$$

- Test statistic for $H_0 : \sigma^2_{\tau\beta} = 0$

$$F_0 = \frac{MS_{AB}}{MS_E} \sim F_{(a-1)(b-1),ab(n-1)}$$

13.3 The Two-Factor Mixed Model

In the mixed model, it is possible to estimate the fixed factor effects as before which are shown here:

$$\hat{\mu} = \bar{y}_{...}$$
$$\hat{\tau}_i = \bar{y}_{i..} - \bar{y}_{...} \quad i = 1, 2, \ldots, a$$

The variance components can be estimated using the analysis of variance method by equating the expected mean squares to their observed values:

$$\hat{\sigma}^2_{\beta} = \frac{MS_B - MS_E}{an}$$
$$\hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_E}{n}$$
$$\hat{\sigma}^2 = MS_E$$

13.3 The Two-Factor Mixed Model

- **Unrestricted mixed models**: no restriction of the random effects terms

$$y_{ij} = \mu + \alpha_i + \gamma_j + (\alpha\gamma)_{ij} + \epsilon_{ijk},$$

where α_i's are fixed effects such that $\sum_i \alpha_i = 0$, $\gamma_j \sim \mathcal{N}(0, \sigma_\gamma^2)$, $(\alpha\gamma)_{ij} \sim \mathcal{N}(0, \sigma_{ij}^2)$, and $\epsilon_{ij} \sim \mathcal{N}(0, \sigma^2)$.

- The expected mean squares

$$\begin{aligned} E(MS_A) &= \sigma^2 + n\sigma_{\alpha\gamma}^2 + \frac{bn\sum_i \alpha_i^2}{a-1} \\ E(MS_B) &= \sigma^2 + n\sigma_{\alpha\gamma}^2 + an\sigma_\gamma^2 \\ E(MS_{AB}) &= \sigma^2 + n\sigma_{\alpha\gamma}^2 \\ E(MS_E) &= \sigma^2 \end{aligned}$$

Subsection 4

13.4 Rules for Expected Mean Squares

13.4 Rules for Expected Mean Squares

An important part of experimental design problem is conducting the analysis of variance.

This involves determining the sum of squares for each component in the model and number of degrees of freedom associated with each sum of squares.

To construct appropriate test statistics, the expected mean squares must be determined.

13.4 Rules for Expected Mean Squares

By examining the expected mean squares, one may develop the appropriate statistic for testing hypotheses about any model parameter.

The test statistic is a ratio of mean squares that is chosen such that the expected value of the numerator mean square differs from the expected value of the denominator mean square only by the **variance component** or the **fixed factor** in which we are interested.

13.4 Rules for Expected Mean Squares

- **Rule 1.** The error term in the model is $\epsilon_{ij\ldots m}$, where the subscript m denotes the replication subscript. For the two-factor model, this rule implies that the error term is ϵ_{ijk}. The variance component associated with $\epsilon_{ij\ldots m}$ is σ^2.

13.4 Rules for Expected Mean Squares

- **Rule 2.** In addition to an overall mean (μ) and an error term $\epsilon_{ij} \ldots m$, the model contains all the main effects and any interactions that the experimenter assumes exist. If all possible interactions between k factors exist, then there are $\binom{k}{2}$ two-factor interactions, $\binom{k}{3}$ three-factor interactions, $\ldots, 1k$-factor interaction. If one of the factors in a term appears in parentheses, then there is no interaction between that factor and the other factors in that term.

13.4 Rules for Expected Mean Squares

- **Rule 3.** For each term in the model, divide the subscripts into three classes:
  - **live** - those subscripts that are present in the term and are not in the parenthesis
  - **dead** - those subscripts that are present in the term and are in the parenthesis
  - **absent** - those subscripts that are present in the model but not in that particular term

E.g. for two-factor fixed effects model, in $(\tau\beta)_{ij}$, i and j are live, and k is absent; in $\epsilon_{(ij)k}$, k is live, and i and j are dead

(We haven't seen models with dead subscripts, but we will encounter such models later.)

13.4 Rules for Expected Mean Squares

- **Rule 4. Degrees of freedom.** The number of degrees of freedom for any term in the model is the product of *the number of levels associated with each dead subscript* and *the number of levels minus 1 with each live subscript*.

E.g. the number of degrees of freedom associated with $(\tau\beta)_{ij}$ is $(a-1)(b-1)$, and the number of degrees of freedom associated with $\epsilon_{(ij)k}$ is $ab(n-1)$.

The number of degrees of freedom for error is obtained by subtracting the sum of all other degrees of freedom from $N-1$, where N is the total number of observations.

13.4 Rules for Expected Mean Squares

- **Rule 5.** Each term in the model has either a variance component (random effect) or a fixed factor (fixed effect) associated with it.

If the interaction term contain at least one random effect, the entire effect is termed is considered as random

A variance component has Greek letters as subscripts to identify the particular random effect, e.g. σ^2_β is the variance component corresponding to random factor B

A fixed effect always represented by the sum of squares of the model components associated with that factor divided by its degrees of freedom, e.g. $\sum_i \tau_i^2/(a-1)$ for factor A when it is fixed

13.4 Rules for Expected Mean Squares

- **Rule 6.** There is an expected mean square for each model component. The expected mean square for error is $E(MS_E) = \sigma^2$.

In case of the **restricted model**, for every other model term, the expected mean square contains

- σ^2 plus
- either the variance component or the fixed effect component for that term, plus
- those components for all other model terms that contain the effect in question and that involve no interactions with other fixed effects.

The coefficient of each variance component or fixed effect is the number of observations at each distinct value of that component.

13.4 Rules for Expected Mean Squares

To illustrate for the case of the **two-factor fixed** effects model, consider finding the interaction expected mean square, $E\left(MS_{AB}\right)$.

- The expected mean square will contain only the fixed effect for the AB interaction (because no other model terms contain AB) plus σ^2, and the fixed effect for AB will be multiplied by n because there are n observations at each distinct value of the interaction component (the n observations in each cell).
- Thus, the expected mean square for AB is

$$E\left(MS_{AB}\right) = \sigma^2 + \frac{n\sum_{i=1}^{a}\sum_{j=1}^{b}(\tau\beta)_{ij}^2}{(a-1)(b-1)}$$

13.4 Rules for Expected Mean Squares

- As another illustration of the **two-factor fixed** effects model, the expected mean square for the main effect of A would be

  $$E\left(MS_A\right) = \sigma^2 + \frac{bn\sum_{i=1}^{a}\tau_i^2}{(a-1)}$$

  The multiplier in the numerator is bn because there are bn observations at each level of A. The AB interaction term is not included in the expected mean square because while it does include the effect in question (A), factor B is a fixed effect.

13.4 Rules for Expected Mean Squares

To illustrate how Rule 6 applies to a model with random effects, consider the two-factor random model. The expected mean square for the AB interaction would be

$$E\left(MS_{AB}\right) = \sigma^2 + n\sigma^2_{\tau\beta}$$

and the expected mean square for the main effect of A would be

$$E\left(MS_{A}\right) = \sigma^2 + n\sigma^2_{\tau\beta} + bn\sigma^2_{\tau}$$

Note that the variance component for the AB interaction term is included because A is included in AB and B is a random effect.

13.4 Rules for Expected Mean Squares

Two factor fixed effects model

$$E(MS_A) = \sigma^2 + \frac{bn\sum_{i=1}^{a}\tau_i^2}{(a-1)}$$

$$E(MS_B) = \sigma^2 + \frac{an\sum_{j=1}^{b}\beta_j^2}{b-1}$$

$$E(MS_{AB}) = \sigma^2 + \frac{n\sum_{i=1}^{a}\sum_{j=1}^{b}(\tau\beta)_{ij}^2}{(a-1)(b-1)}$$

$$E(MS_E) = \sigma^2$$

13.4 Rules for Expected Mean Squares

Two factor random model

$$
\begin{aligned}
E\left(MS_A\right) &= \sigma^2 + n\sigma^2_{\tau\beta} + bn\sigma^2_{\tau} \\
E\left(MS_B\right) &= \sigma^2 + n\sigma^2_{\tau\beta} + an\sigma^2_{\beta} \\
E\left(MS_{AB}\right) &= \sigma^2 + n\sigma^2_{\tau\beta} \\
E\left(MS_E\right) &= \sigma^2
\end{aligned}
$$

13.4 Rules for Expected Mean Squares

Restricted form of two factor mixed model

$$
\begin{aligned}
E\left(MS_A\right) &= \sigma^2 + n\sigma^2_{\tau\beta} + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1} \\
E\left(MS_B\right) &= \sigma^2 + an\sigma^2_{\beta} \\
E\left(MS_{AB}\right) &= \sigma^2 + n\sigma^2_{\tau\beta} \\
E\left(MS_E\right) &= \sigma^2
\end{aligned}
$$

13.4 Rules for Expected Mean Squares

Rule 6 can be easily modified to give expected mean squares for the **unrestricted** form of the mixed model. Simply include *the term for the effect in question*, plus *all the terms that contain this effect as long as there is at least one random factor*.

Unrestricted form of two factor mixed model.

$$\begin{aligned}
E\left(MS_A\right) &= \sigma^2 + n\sigma^2_{\tau\beta} + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1} \\
E\left(MS_B\right) &= \sigma^2 + n\sigma^2_{\tau\beta} + an\sigma^2_{\beta} \\
E\left(MS_{AB}\right) &= \sigma^2 + n\sigma^2_{\tau\beta} \\
E\left(MS_E\right) &= \sigma^2
\end{aligned}$$

Subsection 5

13.5 Approximate F-Tests

13.5 Approximate F-Tests

Consider a three-factor factorial experiment with a levels of factor A, b levels of factor B, c levels of factor C, and n replicates.

First, assume that all the factors are fixed.

$$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \epsilon_{ijkl} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, c \\ l = 1, 2, \ldots, n \end{cases}$$

Then the analysis of this design is given below

13.5 Approximate F-Tests

TABLE 5.12
The Analysis of Variance Table for the Three-Factor Fixed Effects Model

Source of Variation	Sum of Square	Degrees of Freedom	Mean Squares	Expected Mean Square	F_0
A	SS_A	$a-1$	MS_A	$\sigma^2 + \frac{bcn\sum \tau_i^2}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B	SS_B	$b-1$	MS_B	$\sigma^2 + \frac{acn\sum \beta_j^2}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
C	SS_C	$c-1$	MS_C	$\sigma^2 + \frac{abn\sum \gamma_k^2}{c-1}$	$F_0 = \frac{MS_C}{MS_E}$
AB	SS_{AB}	$(a-1)(b-1)$	MS_{AB}	$\sigma^2 + \frac{cn\sum\sum (\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
AC	SS_{AC}	$(a-1)(c-1)$	MS_{AC}	$\sigma^2 + \frac{bn\sum\sum (\tau\gamma)_{ik}^2}{(a-1)(c-1)}$	$F_0 = \frac{MS_{AC}}{MS_E}$
BC	SS_{BC}	$(b-1)(c-1)$	MS_{BC}	$\sigma^2 + \frac{an\sum\sum (\beta\gamma)_{jk}^2}{(b-1)(c-1)}$	$F_0 = \frac{MS_{BC}}{MS_E}$
ABC	SS_{ABC}	$(a-1)(b-1)(c-1)$	MS_{ABC}	$\sigma^2 + \frac{n\sum\sum\sum (\tau\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$	$F_0 = \frac{MS_{ABC}}{MS_E}$
Error	SS_E	$abc(n-1)$	MS_E	σ^2	
Total	SS_T	$abcn-1$			

13.5 Approximate F-Tests

Now, assume that all the three factors are random. The three-factor random effects model is

$$y_{ijkl} = \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} \\ + (\tau\beta\gamma)_{ijk} + \epsilon_{ijkl}$$

Assumptions:

- $\tau_i \sim \mathcal{N}(0, \sigma_\tau^2)$, $\beta_j \sim \mathcal{N}(0, \sigma_\beta^2)$, $\gamma_k \sim \mathcal{N}(0, \sigma_\gamma^2)$
- $(\tau\beta)_{ij} \sim \mathcal{N}(0, \sigma_{\tau\beta}^2)$, $(\tau\gamma)_{ik} \sim \mathcal{N}(0, \sigma_{\tau\gamma}^2)$, $(\beta\gamma)_{jk} \sim \mathcal{N}(0, \sigma_{\beta\gamma}^2)$
- $(\tau\beta\gamma)_{ijk} \sim \mathcal{N}(0, \sigma_{\tau\beta\gamma}^2)$
- $\epsilon_{ijk} \sim \mathcal{N}(0, \sigma^2)$
- All the random effects are pair-wise independent

13.5 Approximate F-Tests

The expected mean squares assuming that all the factors are random are

■ **TABLE 13.8**

Expected Mean Squares for the Three-Factor Random Effects Model

Model Term	Factor	Expected Mean Squares
τ_i	A, main effect	$\sigma^2 + cn\sigma^2_{\tau\beta} + bn\sigma^2_{\tau\gamma} + n\sigma^2_{\tau\beta\gamma} + bcn\sigma^2_{\tau}$
β_j	B, main effect	$\sigma^2 + cn\sigma^2_{\tau\beta} + an\sigma^2_{\beta\gamma} + n\sigma^2_{\tau\beta\gamma} + acn\sigma^2_{\beta}$
γ_k	C, main effect	$\sigma^2 + bn\sigma^2_{\tau\gamma} + an\sigma^2_{\beta\gamma} + n\sigma^2_{\tau\beta\gamma} + abn\sigma^2_{\gamma}$
$(\tau\beta)_{ij}$	AB, two-factor interaction	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + cn\sigma^2_{\tau\beta}$
$(\tau\gamma)_{ik}$	AC, two-factor interaction	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + bn\sigma^2_{\tau\gamma}$
$(\beta\gamma)_{jk}$	BC, two-factor interaction	$\sigma^2 + n\sigma^2_{\tau\beta\gamma} + an\sigma^2_{\beta\gamma}$
$(\tau\beta\gamma)_{ijk}$	ABC, three-factor interaction	$\sigma^2 + n\sigma^2_{\tau\beta\gamma}$
ϵ_{ijkl}	Error	σ^2

- What is the test statistic for $H_0 : \sigma^2_{\tau} = 0$?

13.5 Approximate F-Tests

- For three-factor random effects model, no exact test statistic for testing certain effects, e.g. for $H_0 : \sigma^2_\tau = 0$ one possible test statistic

$$F_0 = \frac{\mathsf{MS}_A}{\mathsf{MS}_{ABC}} = \frac{\sigma^2 + cn\sigma^2_{\tau\beta} + bn\sigma^2_{\tau\gamma} + n\sigma^2_{\tau\beta\gamma} + bcn\sigma^2_\tau}{\sigma^2 + n\sigma^2_{\tau\beta\gamma}},$$

which would be useful if the interactions $\sigma^2_{\tau\beta}$ and $\sigma^2_{\tau\gamma}$ are negligible.

13.5 Approximate F-Tests

- If we cannot assume that the certain interactions are negligible and we need to make inferences about those effects for which exact tests do not exist, Satterthwaite' method can be used.
- Satterthwaite's method uses the **linear combinations of mean squares**, for example

$$MS' = MS_r + \cdots + MS_s$$
$$MS'' = MS_u + \cdots + MS_v$$

are chosen so that $E\left(MS'\right) - E(MS'')$ is equal to a multiple of the effect (the model parameter or variance component) considered in the null hypothesis.

13.5 Approximate F-Tests

Then the test statistic would be

$$F = \frac{MS'}{MS''}$$

which is distributed approximately as $F_{p,q}$, where

$$p = \frac{(MS_r + \cdots + MS_s)^2}{MS_r^2/f_r + \cdots + MS_s^2/f_s}$$

$$q = \frac{(MS_u + \cdots + MS_v)^2}{MS_u^2/f_u + \cdots + MS_v^2/f_v}$$

In p and q, f_i is the number of degrees of freedom associated with the mean square MS_i

E.g.

E.g. For our example, for testing the null hypothesis, $H_0 : \sigma^2_\tau = 0$, we can use the test statistic

$$F = \frac{MS_A + MS_{ABC}}{MS_{AB} + MS_{AC}}$$
$$= \frac{2\sigma^2 + cn\sigma^2_{\tau\beta} + bn\sigma^2_{\tau\gamma} + 2n\sigma^2_{\tau\beta\gamma} + bcn\sigma^2_\tau}{2\sigma^2 + 2\sigma^2_{\tau\beta\gamma} + bn\sigma^2_{\tau\gamma} + cn\sigma^2_{\tau\beta}}$$

E.g.

- Under H_0, the statistic F follows F-distribution with p and q degrees of freedom, where

$$p = \frac{(MS_A + MS_{ABC})^2}{(MS_A^2/f_A) + (MS_{ABC}^2/f_{ABC})}$$
$$q = \frac{(MS_{AB} + MS_{AC})^2}{(MS_{AB}^2/f_{AB}) + (MS_{AC}^2/f_{AC})}$$

and f_A is degrees of freedom associated with MS_A.